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ALGEBRA SELF-TAUGHT.

BY

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INTRODUCTORY.

Numerous attempts have been made to interest the general reader in the science of mathematics. Some of these attempts have succeeded; others, and the major portion, have failed. A careful study of the cause of failure leads to the observation of a striking similarity in the unsuccessful attempts; and this similarity lies in dealing to excess with abstract ideas. Mathematics is a tool invaluable when applied to the shaping of some end; but it is possible, even upon grinding, sharpening, or adjusting a tool, to waste time. Indeed, in the cutting of rough wood an edge may be too keen. So it is with mathematics; and the reader who wishes to progress may be made to feel that he is constantly at the grindstone. He may admit that by pre-eminence mathematics is the science of abstraction; but experience has taught him that the greatest good has arisen from its application to concrete methods.

Nor would it be difficult to show that mathematicians, by an excessive devotion to the science in its purity (the proper term for its fullest abstraction), have rendered it unpopular. In its transcendental dress the student approaches mathematics with awe; and, if bold enough to question, receives its teachings, as Professor Airy has aptly said, "rather with the doubts of imperfect faith than with the confidence of rational conviction." In his 'Principles of the Differential Calculus,' Mr. Woolhouse observes,

with only too much reason, how "it is to be regretted that most of our academical treatises on this, as well as other subjects, abound so much with complex algebraic processes, without the slightest traces of logical reasoning to exercise and improve the intellect. We should bear in mind that the simple execution of analytical operations acquired by dint of practice and experience is a mere common species of labour, often merely mechanical; whilst a distinct apprehension of the specific object and meaning of the operations, and a contemplation of the clearness and beauty of the various arguments employed, constitute the intellectual lore that gratifies and enriches the mind, and stimulates its energies with an ardour after the investigation of truth."

Mathematics naturally divide into two divisions, to which are related (α) the solutions of problems where it is the object to ascertain the value of unknown (from the relation which they bear to known) and constant quantities; and (β) the investigation of variable quantities increasing or decreasing uniformly with regard to other quantities that are constant. When dealing with (α) we are in the dominions of ordinary *Algebra;* the consideration of β is effected by means of the Differential and the Integral Calculus. In the present work, then, we have only to take into view *constant* quantities, or quantities to which, if a value is once assigned, retain that value to the end of the problem.

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CHAPTER I.

SYMBOLS AND THE SIGNS OF OPERATION.

In arithmetic we have learned to consider two kinds of numbers, *abstract* and *concrete*. 1, 2, 3, or 4, etc., is an abstract number, because it denotes an abstract idea of how many times; but 1 apple, 2 apples, or 3 apples, etc., is a concrete number, because it denotes so many actual substances. We cannot see or feel 2 as a number until it is embodied in the form of the two apples.

In algebra, however, the division into concrete and abstract need not be a source of trouble; for we may substitute a *letter* (a, b, c . . . . . x, y, z) for either an abstract or concrete number. Thus a may be put to represent 1 or 1 apple, or 2 or 2 apples. If a be taken as a kind of short way of writing one apple, $2a$ will stand for two apples; but if a is taken to represent two apples, then $2a$ will stand for four apples; and so on. Similarly, if a be taken to represent the figure 1, then $2a$ will be equal to 2. If a stand for 2, then $2a$ will equal 4.

This is enough, perhaps, to enable the student to understand the definition—*Algebra is the Science of Computation by Symbols.* The symbols generally employed are the letters of the alphabet, sometimes those of the Greek

alphabet. There is no reason why any system of symbols, however arbitrary, should not have been used; but universal custom has given the preference to letters, arbitrary symbols being employed when the number of things or of abstract ideas to be represented exceeds the number of letters.

It will be convenient to remark that there is in common use in algebra a sign for equality, or for the words *equal to.* It is $=$. If we wish to express that one and one make or equal two, we can write 1 and $1 = 2$.

That also there is a sign for addition, called *plus,* and it is $+$. So that we can state shortly 1 added to $1 = 2$, or $1 + 1 = 2$. Again, there are signs for subtraction, multiplication, and division, to which we must presently refer.

There is some difference between the operations of arithmetic and those of algebra. Instance may be taken in adding 2 and 3. Arithmetically $2 + 3 = 5$; but if we place a for 2 and b for 3, we can only give the result in algebra as $a + b$, unless we previously arrange that c shall equal 5. In the last case we can say $a + b = c$, one of the simplest forms of an algebraic equation. Let us quite understand this. If we have a basket of fruit containing apples (call them a), and blackberries (call them b); we cannot say *apples* + *blackberries* equal *pears;* we can only give the result as *fruit,* which is a general and not a particular term. Consequently we must be satisfied with the statement of the contents of the basket (if we wish to describe particularly) as being apples and blackberries. Similarly, in algebra, we often cannot describe a more definite result than $a + b$.

What does this negative evidence teach? It shows that the sign $+$ is a sign indicating that the action of

addition is to be performed—is, in fact, a sign of operation. So are all the signs in algebra. They indicate *what is to be done* when we have found the values of the symbols or letters which they connect. A complete formula, indeed, is nothing more than a *symbolic* direction to perform certain actions, to attain a desired result, which result may or may not be expressed.

CHAPTER II.

THE EQUATION AND THE UNKNOWN QUANTITY.

We have shown how the addition of 3 to 2 and the sum obtained may be shortly written, viz. $3 + 2 = 5$. This (again taking a for 2, b for 3, and c for 5), it has also been shown, may be represented algebraically as

$$a + b = c.$$

From the *sign of equality* (=) connecting the two members, all similar equal quantities so connected are classed under the general term of an *equation*. The *solution of an equation*, or the *solving* of an equation, merely means the working of it out until the *answer* has been found. The *answer* is the *unknown quantity*.

The unknown quantity is generally represented by the letter x, or, if there are more than one, by the letters x, y, z (known quantities, as we have said, being represented by the letters a, b, c . . . . . etc.).

Substituting, therefore, the letter x for the letter c in the foregoing equation, we have

$$a + b = x \text{ (or the answer).}$$

As we have determined that a shall equal 2 and b shall

equal 3, x, or the unknown quantity, or the answer, must equal 5. Thus we have as synonymous terms

$$a + b = x,$$

or

$$2 + 3 = 5.$$

Now we may so ring the changes on the two members of the equation as to form *new* equations. We say *new*, because for *two equations to be equal* (*or to be equated*) *their results* (*or their x's*) *must be equal.*

To test this, let us add 2 to *both* sides of the equation. We have

$$2 + 3 + 2 = 5 + 2, \text{ or } 7.$$

Evidently a new equation, because 5, the former result (or x), does not equal 7. Similarly, if we multiply both sides by 2, or divide them by 2, or take 2 from them, we shall obtain new equations. We learn, then, that the balance of an equation is not destroyed when we treat in any way *both* sides by the same quantity. This is the characteristic of an equation, and from this characteristic there springs a valuable aid to the solution of (or finding the answer to) equations.

Let us try to understand its application.

There is a sign employed in algebra (and sometimes in arithmetic) indicating *subtraction* or *taking from:* the sign is called *minus*, and is $-$. So that we may state the equation, $3 - 2 = 1$, or $b - a = x$. Suppose we are told (adhering to our former values for a and b) that

$$a = x - b,*$$

* The student will understand that I am dealing with these simple quantities in order that he may not have to think of the actions of his mind; for he will presently perceive that what is true of, and can be done with, these simple quantities, is as equally true of others, however complex.

and that we are to find the numerical value of x. How should we set about doing it?

Evidently we should say, substituting figures, that

$$2 = 5 - 3,$$

or, what is the same thing, $5 - 3 = 2$. But we wish to know what x is equal to, and to know this it is evident that all we have to do is to follow our direction implied in the equation, to take 3, or b, from the right side of the equation, and add it to the left, and so get back our old form

$$a + b = c, \text{ or } 2 + 3 = 5.$$

Or we may proceed in another way, which is, mathematically, the more accurate. We may add 3, or b, to both sides of the equation, thus:

$$a + b = x - b + b,$$

or

$$2 + 3 = 5 - 3 + 3,$$

and as subtracting 3 and adding 3 result in nothing, we have again $a + b = x$. This operation of taking from one side of an equation to add to the other, or *vice versâ*, the equality being preserved, is termed *transposition.*

We ask our students to set to work and repeat this process for themselves with other letters and other numbers. They will derive much greater benefit from such practice than from poring over the unpractical work even now often given in our text-books.*

* I will make clear what I mean by quoting my lamented friend and former master, Professor de Morgan, who had vast dislike to the then too common method of illustrating the teaching of algebra by examples of *useless* equations. "In answer," he says, "to a defence sometimes set up, that the system is *practical*, we observe that much of what is done has no reference to any practical end whatever. The great body of the algebraical work of a school consists in questions of multiplication and division which never occur in practice; above all, in the

In our next chapter we come to the consideration of *negative* quantities, or those quantities erroneously valued as "less than nothing."

CHAPTER III.

POSITIVE AND NEGATIVE QUANTITIES.

HITHERTO we have been dealing with *positive* quantities—that is, with those quantities which are arrived at by counting upwards from zero or 0. But let us imagine ourselves counting *downwards* from zero or 0, and we arrive at so many "less than nothing." "Less than nothing," however, as I shall presently endeavour to show, is an incorrect expression. It is convenient to call those

solution of certain *conundrums* called *problems*, producing equations, of the *practical* nature of which the reader shall judge from the following specimens:

"'A person being asked the hour, answered that it was between five and six, and the hour and the minute hands were together. What was the time?'

"'A post is one-fourth in the mud, one-third in the water, and 10 feet above the water; what is its whole length?'

"'A person has two horses and a saddle worth 50*l.*; now if the saddle be put on the back of the first horse it will make his value double that of the second, but if it be put on the back of the second it will make his value triple that of the first. What is the value of each horse?'

"Now if all this be meant for improvement in theory, no one will deny that the reasons of all the rules should be previously understood; but if they be practical questions, we need only say that people have more pertinent methods of answering the question, 'What's o'clock?'—that no one concerns himself about the proportion in which a post is shared between wind, water, and mud—and that the Newmarket gentry have a better way of determining the value of their horses than by involving them, saddles and all, in a simple equation."

Far be it, however, that I should condemn the general system of teaching by books; I wish merely to encourage the disheartened student who may have succumbed not for want of energy, but its waste.

quantities above 0 *positive*, and those below 0 *negative*. Positive quantities may or may *not* have the positive sign (+) prefixed to them, but negative quantities have always the negative or minus sign (−) prefixed. $+a$ is a positive quantity; $-a$ is a negative quantity.

In order to perfectly understand the distinction between positive and negative quantities, we may consider two illustrations.

I have a thermometer, and I plunge it into warm water. The instrument indicates say 20° of heat, and this may be written + 20°. Or I plunge the thermometer into a freezing mixture, and the mercury sinks below zero, say 20°; this may be written − 20°. 20° of heat is then written + 20°, and 20° of cold is written − 20°. Now 20° of cold is not 20° "less than nothing"; it is a *want* of 20° of heat to bring about the normal or zero condition.

The second example will be still more familiar. I *have* three apples, and I may write this as + 3 or $+ 3a$; I *want* three apples, and I may write this as − 3 or $- 3a$. In the latter case it will take three apples to satisfy my want, and another three apples to make me rich in the amount of three apples.

Thus, $- 3a$ does not mean three apples "less than nothing," which would be an absurdity, but it means a *want* of three apples to restore my appetite for the fruit to its normal state, or state of no desire.

The expression or equation, it will be seen, holds good that

$$-3a + 6a = +3a,$$

for if we take $6a$ as composed of two $3a$'s, it will absorb one $3a$ to satisfy the want expressed by the negative or minus sign, and we shall have one $3a$ remaining.

If the reader have mastered the distinction between

positive and negative quantities he has passed the *pons assinorum* of algebra. If he be in possession of the correct meaning, he will understand that the distinction consists in measuring in two different directions,—that is, if by $+$ he is told to measure to the right, by $-$ he knows he is to measure to the left. The signs indicate opposition,—to add to and to take from,—the possession, and the want of.

That the reader may be assured of his comprehension let him resolve for himself the following equations:

$$-20+10=-10,$$
$$10-20=-10.$$
$$-2p+6p-4p=0,$$
$$-4p+6p-2p=0,$$
$$6p-2p-4p=0.$$

The last expression leads to the consideration of the mathematical use of *brackets*. Brackets in mathematics are similar in form to the brackets

( ) [ ] { }

of print and writing, but they do not, as those in print, indicate inferiority or illustration. Mathematically they are signs indicating *collection*. Thus, in the expression

$$6p-2p-4p=0,$$

the repetition of subtraction indicated by $-2p-4p$ would be equally well represented by $-(2p+4p)$, or by adding the two negative quantities and taking their sum from $6p$, with the result $=0$. Brackets are employed to class or collect together several *simple* quantities which may be taken together as one *compound* quantity. Instead of brackets, the *vinculum*, a line drawn above the com-

pound quantity, is sometimes used. Thus the following are identical expressions:

Brackets.	Vinculum.
$6p - (2p + 4p) = 0.$	$6p - \overline{2p + 4p} = 0.$

Brackets sometimes occur within brackets, and then the following arrangement is most general:

$$12p - [6p + (2p + 4p)] = 0.$$

Though simple the reasoning we have followed, the rule for subtraction may be stated even more simply:

Change the signs of all the quantities to be subtracted, and then add.

Let us test this numerically:

1. By subtracting a positive number from a positive number, as $4 - 2 = 2$, and $4 + (-2) = 2$; or rendering a positive number less positive, for in the latter case it takes 2 of the positive number 4 to satisfy the want expressed by the negative sign prefixed to the 2.
2. By subtracting a negative number from a positive number, as

$$4 - (-2) = 6,$$

which is simply a direction to perform the operation of subtracting or removing the *want* of 2, expressed by (-2), from the 4. We may consider this as a direction to increase a positive number, the direction being written under a negative form.

3. By subtracting a negative number from another negative number, as $-4 - (-2) = -2$, or, in

other words, removing the want expressed by the negative quantity (-2).

4. By subtracting a positive number from a negative number, as $-4-2=-6$, or rendering a negative quantity still more negative.

Let us regard the question, which is one of great difficulty, in another light. We know that 4 may be written $4+2-2$; for as $2-2$ equal nothing, the addition of 0 to 4 does not alter its value. Let us consider subtraction as meaning "taking away," and, starting with $4+2-2=4$, let us see the effect of taking away first the $+2$, and then the -2, which is simply a repetition of the examples given above in subtracting a positive or a negative number from a positive number. Thus:

SUBTRACTION FROM A POSITIVE NUMBER.

	4	+ 2	− 2	= 4
(1)	4	*	− 2	= 2
(2)	4	+ 2	†	= 6

Similarly we may illustrate the two examples remaining, by considering -4 as written $-4+2-2$, where, as before, $2-2$ is nothing.

SUBTRACTION FROM A NEGATIVE NUMBER.

	− 4	+ 2	− 2	= − 4
(3)	− 4	+ 2	†	= − 2
(4)	− 4	*	− 2	= − 6

* Taking away the positive number.
† Taking away the negative number.

It will perhaps prevent misconception if the reader considers the omissions in print indicated by the signs * and † as representative of rubbing out from a slate the corresponding numbers. If the numbers are put down upon a slate, and instead of being taken away are rubbed out, the process may be still more clearly understood.

We can, from the preceding, perceive the reason for the following rules:

Rules for Brackets.

If the sign + (*plus*) be before the bracket, the brackets may be removed without altering the value of the expression.

If the sign – (minus) *be before the bracket, remove this sign* WITH *the brackets, and* CHANGE *all the signs of the quantities within the brackets.*

CHAPTER IV.

MULTIPLICATION, INVOLUTION, EXPONENTS.

UNDER this triple head we have to consider three subjects usually dissevered, but which really arise from one operation, that of multiplication.

Multiplication.—Early in our studies we understood that if a be taken as a kind of short way of writing one apple, $2a$ will stand for two apples; but if a is taken to represent two apples, then $2a$ will stand for four apples; and so on.

Similarly, if a be taken to represent the abstract number one, then $2a$ will be equal to 2. If a stand for 2, then $2a$ will equal 4. Thus the figure 2 is a *multiplier*, and

all multipliers in similar position are termed *coefficients.* In the instance given the coefficient of a is 2, a numerical coefficient.

Coefficients are *numerical* and *literal.* If it is understood that n shall represent any determined number, then $n\,a$ will represent n times a, and n will be the literal coefficient of a.

It is convenient to indicate the operation of *multiplication* by letters placed together, as $n\,a$. This operation is indicated also by the sign $\times$ (read "into" or "multiplied into") between two quantities, as $a \times b$, or by a point . *, as $a\,.\,b$. Thus $a\,b$, $a \times b$, and $a\,.\,b$ are identical in meaning.

It will at once be seen how we may indicate multiplication of a bracketed quantity, as

$$2\,(a + b) = 2\,a + 2\,b,$$

or

$$n\,(a + a\,b) = n\,a + n\,b;$$

and how we may obtain a rule for the removal of the brackets when a coefficient stands before them,—that is, *when removing brackets, to multiply all the terms of the bracketed quantity by the coefficient.*

As instances, the reader may trace out the meaning of the following equations:

$$3\,a\,b\,(x + y) = 3\,a\,b\,x + 3\,a\,b\,y.$$

$$n\,(a\,x + b\,y) = n\,a\,x + n\,b\,y.$$

$$a\,(b + c) \times (d + e) = (a\,b + a\,c) \times (d + e) = a\,b\,d + a\,b\,c + a\,c\,d + a\,c\,e.$$

We have learned that a *coefficient* is a *multiplier.* It is a common error, arising from want of accuracy in limiting

* To prevent confusion with decimal fractions the dot is not employed in numerical calculation.

the meaning of words, to call a constant quantity that is to be added, subtracted, or to be a divisor, by the name of a coefficient. Such a quantity is more conveniently termed a *constant*. In the expression $x + c$, $x - c$, c is a constant; in the expression $c\,x$, c is a coefficient.

The idea of multiplication includes the multiplication of quantities having like and unlike signs,—that is, we may be called upon to perform the operation of multiplying (1) a positive quantity by a positive quantity, (2) a negative quantity by a positive quantity, and (3 and 4) the converse.

The multiplication of a positive quantity by another positive quantity will clearly result in a positive product.* But the result of multiplying together two negative quantities, or a negative and positive quantity, is not so apparent. Let us look more closely at the question.

There are four possible cases of multiplication, viz.:

$$
\begin{array}{ll}
(1) & a \times b. \\
(2) & -a \times b. \\
(3) & -a \times -b. \\
(4) & a \times -b.
\end{array}
$$

The first two are easily understood. In the first case we take a the number of times represented by b; in the second case we take $-a$, b times. That is, if a is equal to 2, and b represents 3, then $a \times b$ is equal to 6; and $-a \times b = -6$, or -2 repeated three times.

In the fourth case, where a is to be multiplied by $-b$, we are directed by the sign of operation to repeat a the number of times represented by $-b$, and as we may

* The result of multiplying two quantities is termed the *product*; the result of their addition, the *sum*.

always reverse the process * and repeat $-b$ the number of times represented by a, the product is again -6.

Thus, in both cases where the quantities have *unlike* signs, the product is a minus or negative quantity.

The remaining (third) case of like signs (that of multiplying two negative quantities) it is not so easy to grasp. We must arrive at the result indirectly. The question, numerically stated, is to multiply -2 by -3. If we increase the multiplier (-3) by 4, making it equal $+1$, the product $(=-2)$ will, of course, be four times too great, and to arrive at the true result we must subtract 4 times -2, or -8. Now $-2-(-8)$ is, as we have seen when studying subtraction, equal to $+6$.

We have thus arrived at the conclusion, which may be stated as a rule, that the multiplication of quantities having like signs gives a positive product, and the multiplication of quantities having unlike signs gives a negative product. The rule is concisely stated as follows:

Like signs give PLUS; *unlike signs,* MINUS.

Involution and Exponents.—If we desire to multiply a by itself, we may state the result as $a\,a$, and a repetition of the operation would give $a\,a\,a$. Custom has, however, established a more convenient expression than $a\,a$ or $a\,a\,a$, namely, a^2 or a^3.†

* In all calculi, with the exception of Hamilton's 'Quaternions' and of De Morgan's 'Double Algebra,' we may say that $a\,b\,c = a\,c\,b = b\,c\,a = b\,a\,c = a\,c\,b$, or that $a\,b = b\,a$, and this is true where a, b, and c represent numerical *magnitude* only. When they represent (as in the calculi quoted as exceptions) *direction* as well as magnitude, their change or commutation changes the sign of the product. In common algebra the commutation is without this effect.

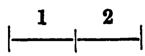
† The reader must not confound this expression with $2\,a$ or $3\,a$. a^2 means a times a; a^3, a times a times a; but $2\,a$ or $3\,a$ means twice or thrice a.

The small figure ^{2} or ^{3} is termed the *exponent*. An exponent shows the number of times a quantity is to be multiplied by itself, or, in technical phraseology, the "power to which the quantity is to be raised." The first power of a given quantity, or a^1, is equal, of course, to the quantity itself. The second power of a given quantity, or a^2, is termed the "square" of that quantity. The third power is termed the "cube." Other powers are known by their ordinal names.

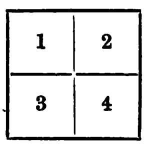
If the exponent be *fractional* (that is, not an integer or whole number) it indicates a "*root*," a matter we shall presently describe.

The term "square" is derived from the fact that if we wish to arrive at the superficial contents, or area expressed in square measure of a given (rectangular) space, we may, by multiplying the linear measures of the (two) sides of the space, obtain this area; and if again we multiply this area by the given quantity, we have the expression in cubic measure of the solid contents of the space. When referring to geometrical expressions, we should remember that if we represent a line by a, the square built upon that line will be a^2, the solid or cube built upon the square will be a^3.

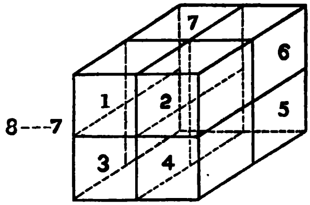
Thus, upon a line of two units' length



if we build a square, we have, as result, four square units,



and if upon this square we erect a cube, we have eight cubic or solid units,



Geometricians give to the line, the square, and the cube, the term of "space of one, two, and three dimensions," respectively. Space of a fourth dimension would be practically inconceivable.

a may then represent space of one dimension,			
a^2	„	„	two dimensions,
a^3	„	„	three dimensions.

To be familiar with these ideas, which are really the result of a false analogy, will be necessary to the understanding of the higher mathematics; we pass now to the consideration of Exponents as Logarithms.

CHAPTER V.

NEGATIVE EXPONENTS, ROOTS, AND THE USE OF EXPONENTS AS LOGARITHMS.

We have learned that algebraists agree to write, for the multiplication indicated by $x\,x$ or $x\,x\,x\,x$, the expression x^2 or x^4.

$$x\,x + x\,x\,x\,x \text{ and } x^2 + x^4,$$

or

$$x\,x - x\,x\,x\,x \text{ and } x^2 - x^4,$$

is the simplest manner in which we can express the addition and subtraction of these quantities. Now let us find out for ourselves the effect, upon the exponents, of the multiplication and division of the quantities x^2, x^4. When we multiply x^2 by x^4 the product shows us a wonderful property pertaining to the exponents.*

Thus,

$$x\,x \,.\, x\,x\,x\,x = x\,x\,x\,x\,x\,x, \text{ or } x^6,$$

or

$$x^2 \times x^4 = x^6\,;$$

whence we see that

$$x^2 \times x^4 = x^{2+4} = x^6,$$

or, in words, that

Multiplying the quantities x^2 and x^4 together is equivalent to adding their exponents or indices.

If we divide† x^4 by x^2 we shall see that $x\,x\,x\,x$ contains $x\,x$ the number of times represented by $x\,x$; that is, $x\,x$

* Index is another term for exponent; plural, Indices.

† Division (upon which we shall presently enter fully when we are in a position to discuss its correlations) is represented algebraically and arithmetically by the sign $\div$, or more generally by writing the quantity

must be multiplied by $x x$ to produce $x x . x x$ or $x x x x$; and this is but saying that x^4 divided by x^2, or $x^4 \div x^2$, or $\frac{x^4}{x^2} = x^2$. We learn, then, that

Dividing the quantity x^4 by x^2 is equivalent to the subtraction of the index (²) *of the divisor from the index* (⁴) *of the dividend;* or, in symbols,

$$\frac{x^4}{x^2} = x^{4-2} = x^2.$$

Now this subtraction of indices naturally includes the extension of the idea to *negative* indices.

Negative Indices or Exponents.—If we reverse the position of the quantities given above, we obtain a result with a negative index,

$$\frac{x^2}{x^4} = x^{2-4} = x^{-2}.$$

Again, by extending our view of the meaning of the term index or exponent, we may arrive at some remarkable, and, as we shall presently perceive, very useful results.

x and x^1 we know to be identical in value, so that

$$\frac{x^1}{x^1} = \frac{x}{x}.$$

But

$$\frac{x^1}{x^1} = x^{1-1} = x^0$$

---

to be divided above a short line, and the divisor below the line, as in fractions. 1 divided by 2 may be written $1 \div 2$, or $\frac{1}{2}$; that is, one-half. Similarly a divided by b is written $\frac{a}{b}$ or $a \div b$. The result of the division is termed the quotient.

and

$$\frac{x}{x} = 1,$$

whence

$$x^0 = 1$$

(which expression, if the reader will presently return to it, he will perceive to be the reason or proof that the logarithm of unity to any base is zero; and since $a = a^1$ the logarithm of the base of any system is unity).

Following our reasoning, we may say

$$\frac{x^0}{x^1} = x^{0-1} = x^{-1};$$

but substituting for x^0 the value we have just obtained (namely, $x^0 = 1$), we have

$$\frac{x^0}{x^1} = \frac{1}{x^1}.$$

Now as

$$\frac{x^0}{x^1} = x^{-1} \text{ and is also equal } \frac{1}{x},$$

$$x^{-1} \text{ must equal } \frac{1}{x^1};$$

or, in other words,

As unity (1) divided by any quantity is the reciprocal of that quantity; $\frac{1}{x}$ is the reciprocal of x; $\frac{1}{2}$ is the reciprocal of 2; then *a quantity raised to a negative power is equal to the reciprocal of the positive power of that quantity;* or, in symbols,

$$x^{-n} = \frac{1}{x^n},$$

whatever number n may stand for. By which we may arrive at the value of a negative power.

Roots.—As we can have integral or whole positive and negative exponents, so an extension of reasoning will comprehend fractional exponents.

Fractional exponents are employed to indicate *roots.*

A *root* is the quantity that produces a *power.* As the second power or square of 2 is 4, so the square root of 4 is 2. As the third power or cube of 2 is 8, so the cube root of 8 is 2.

The radical sign $\sqrt{\ }$ (a corrupt form of the letter *r*, the initial letter of the word "radix" or "root") is employed to indicate the lowest or square root of a quantity. And similarly $\sqrt[3]{\ }$ is employed to indicate the cube root, $\sqrt[4]{\ }$ the fourth root, etc. Thus

$$\sqrt[2]{4} \text{ or } \sqrt{4} = 2$$
$$\sqrt[3]{8} = 2$$
$$\sqrt[4]{16} = 2$$

Or, substituting for the radical signs fractional exponents, we have

$$4^{\frac{1}{2}} = \sqrt{4} = 2$$
$$8^{\frac{1}{3}} = \sqrt[3]{8} = 2$$
$$16^{\frac{1}{4}} = \sqrt[4]{16} = 2$$

It is immaterial whether we read $\frac{1}{2}$ as "the square root of the first power," or as "the (first) power of the second or square root," for the result is the same, namely, 2. Also

$$x^{\frac{n}{n}} = x,$$

which is but raising a quantity to a power to reduce it again to its radical or root value.

Resumé.—As the reader may find it convenient to recur to the principles to which he has advanced in this

chapter, we state in the following *resumé* the deductions from our reasoning.

With powers of the same quantity:

To MULTIPLY, *add* the indices.

To DIVIDE, *subtract* the index of divisor from index of dividend.

To find a POWER, *multiply* the index of the quantity by 2 or 3 . . . . . or (n) the number of the required power.

To find a ROOT, *divide* the index of the quantity by 2 or 3 or (n) the number of the required root.

Now since every quantity (a or x) may be written under the form a^1 or x^1, these rules will hold good for all quantities, and admit of the general expressions

$$x^1 \times x^1 = x^{1+1} = x^2$$

$$\frac{x^1}{x^1} = x^{1-1} = x^0;$$

$$\frac{x}{x} = 1 \text{ and } x^0 = 1$$

$$x^2 = x^{1\times 2} \text{ (or the index 1 multiplied by 2).}$$

$$\sqrt[2]{x} = x^{\frac{1}{2}} \text{ (or the index 1 divided by 2).}$$

The practical reader may ask, What is the benefit of all this? It is, that by a very simple application of these principles are our sailors enabled more easily to navigate our ships; the naval architects to design them; the engineer to build our bridges, our locomotives; the telegraphist to construct and lay our cables; the astronomer to tell us of the movement of other worlds. The application is *Logarithms*, without which our laboratories, studies, and workshops would indeed be badly off.

CHAPTER VI.

LOGARITHMS.

The term *logarithm* is derived from λογων αριθμος (logon arithmos), or the *number of ratios.* Thus between 1 and 10 there is one ratio; between 1 and 100, two ratios; and between 1 and 1000, three ratios of 1 to 10. The definition has been given that "The logarithm of a number (to a given base) is the index of the power to which the base must be raised to give the number."

Thus if $n = b^l$, l is the logarithm of n, the given number to the base b. And if $n = 100$, and b, the base, be 10, then $l = 2$; for

$$100 = 10^2.$$

If $n = 1000$, then $l = 3$; for

$$1000 = 10^3; \text{ and so on.}$$

In the following table let n = the number for which we wish to find the logarithm; b, the base; and l, the index or logarithm, and we have

n		b	l
1	=	10	0
10	=	10	1
100	=	10	2
1000	=	10	3

That is, the logarithm of 1 is 0,
10 is 1,
100 is 2,
1000 is 3, and so on.

Consequently, for all numbers between 1 and 10 the logarithm or index is a fraction: between 10 and 100, 1 plus a fraction; between 100 and 1000, 2 plus a

fraction. These fractions have been calculated, and are to be found in tables, of which we shall presently give an example.

The logarithm of a number, n, to a base b, is usually written $\log._b\, n$; and so if $n = b^l$,

$$l = \log._b\, n.$$

If the base b were 2, and the number 8, the logarithm would be 3, and the expression would become

$$3 = \log._2\, 8.$$

But, as our notation is founded upon the decimal system, the base of the common system of logarithms is 10, and this being understood, the letter b is omitted. Thus the logarithm of 10 is written log. 10 = 1.

Before we proceed to the indices of intermediate numbers, we will first apply our deductions to the numbers with which, as logarithms, we are already acquainted.

Logs.	0	1	2	3	4	5	6
Nos.	1	10	100	1000	10,000	100,000	1,000,000

Rule I.—The sum of the $\left\{\begin{array}{c}\text{logarithms}\\ \text{exponents}\end{array}\right\}$ of two numbers is equal to the $\left\{\begin{array}{c}\text{logarithm}\\ \text{exponent}\end{array}\right\}$ of their product.

Rule II.—The difference of the $\left\{\begin{array}{c}\text{logarithms}\\ \text{exponents}\end{array}\right\}$ of two numbers is equal to the $\left\{\begin{array}{c}\text{logarithm}\\ \text{exponent}\end{array}\right\}$ of the quotient * of these numbers.

Rule III.—The $\left\{\begin{array}{c}\text{logarithm}\\ \text{exponent}\end{array}\right\}$ of the nth power of a

* Or result of division.

number is equal to n times the $\left\{\begin{array}{c}\text{logarithm}\\ \text{exponent}\end{array}\right\}$ of the number.

RULE IV.—The $\left\{\begin{array}{c}\text{logarithm}\\ \text{exponent}\end{array}\right\}$ of the nth root of a number is equal to the nth part of the $\left\{\begin{array}{c}\text{logarithm}\\ \text{exponent}\end{array}\right\}$ of the number.

These rules comprehend the entire use of logarithmic tables, and we will now proceed to their application.

Example to Rule I.—

$$\begin{array}{lr} \text{Log.} & 100 = 2 \\ \text{Log.} & 1000 = 3 \\ \hline \text{Log.} & 100{,}000 = 5 \end{array} \qquad 100 \times 1000 = 100{,}000$$

Example to Rule II.—Log. 100 = 2. Log. 100,000 = 5. Five less two = 3, and three is the log. of 1000; 100,000 divided by 100 gives 1000.

Example to Rule III.—The third power of 10 is 1000. Three times the log. of 10 (or 3×1) is the log. of 1000.

Example to Rule IV.—The cube root of 1000 is 10. The result of dividing the log. of 1000 (or 3) by 3 is 1, the log. of 10.

The examples will appear to the student to be self-evident, and it may be thought that logarithms afford only a tedious method of doing what might be more easily done by other rules. But it should be remembered that in practice far more difficult numbers occur than it would be well to select for illustration.

Having mastered the use of logarithms as integral numbers, we have to deal with the logarithms of those numbers intermediate to 1 and 10, 10 and 100, 100 and 1000, etc. The logarithms of those numbers between 1 and 10 must, it has been shown, be a fraction less than 1,

because the log. of 10 is only 1. Between 10 and 100 the log. of the intermediate numbers is 1 plus a fraction; between 100 and 1000 the log. is 2 plus a fraction, and so on; because 10 (the base) must be raised to the second-plus-a-fraction-power to give a number higher than 100.

The integral portion of the logarithms of higher numbers than 10 is termed the *characteristic* of the logarithm, and the fractional portion is termed the *mantissa.**

In logarithmic tables generally the characteristics are omitted because it is easy for the calculator to find them by the obvious rule that "*the characteristic of the logarithm of a number is always* ONE LESS *than the number of integral figures of the number.*" This will be seen in the following table of logarithms of numbers from 1 to 20, whence if we take the log. of 19 (which is 1·27875), we shall find the characteristic † is one less than the number (2) of integers contained in 19.

TABLE OF LOGARITHMS OF NUMBERS FROM 1 TO 20.‡

No.	Log.	Arithmetical Complement.	No.	Log.	Arithmetical Complement.
1	0·00000	0·00000	11	1·04139	0·95861
2	0·30103	0·69897	12	1·07918	0·92082
3	0·47712	0·52288	13	1·11394	0·88606
4	0·60206	0·39794	14	1·14613	0·85387
5	0·69897	0·30103	15	1·17609	0·82391
6	0·77815	0·22185	16	1·20412	0·79588
7	0·84510	0·15490	17	1·23045	0·76955
8	0·90309	0·09691	18	1·25527	0·74473
9	0·95424	0·04576	19	1·27875	0·72125
10	1·00000	0·00000	20	1·30103	0·69897

* A Latin word, meaning an additional handful, something over and above; plural, mantissæ.

† The characteristic is also termed the index.

‡ We do not include in these lessons an extended or practical table of logarithms, because such a table would absorb too much space, and is rendered unnecessary by the many excellent pocket and other tables published at a low price.

The student will notice that the log. of 20 is, as far as the mantissa is concerned, the same as the logarithm for 2, and this arises from the fact that 2 bears the same ratio to 10 as 20 does to 100. Similarly the log. of 200 is $2 \cdot 30103$; of 150, $2 \cdot 17609$. He will notice, too, the column headed Arithmetical Complement. The arithmetical complement of a log. is the difference of the mantissa of that log. from unity. The use we shall describe presently. Let us now apply to various examples our four rules previously given.

Rule I. *Example.*—To multiply 90 and 120.

$$\begin{aligned} \text{Log. } 90 &= 1 \cdot 95424 \\ \text{Log. } 120 &= 2 \cdot 07918 \\ \hline \text{Log. } 10800 &= 4 \cdot 03342 \end{aligned}$$

A result which the student may verify by reference to a table.

Rule II. *Example.*—To divide 418 by $2 \cdot 4$. The log. of $2 \cdot 4$ is $0 \cdot 38021$, the characteristic being 0, because there is only one integer in the number; but the student looks for the log. of 24 in the table.

Log. $418 = 2 \cdot 62118$. The difference between the two logarithms is $2 \cdot 24097$, the log. of $174 \cdot 16$.

It is a law that (as we shall see when we study the subject of division) to divide by any number n is productive of the same result as to multiply by its reciprocal $\frac{1}{n}$. So it follows that to subtract log. n produces the same result as to add log. $\frac{1}{n} = 0 - \text{log. } n$. Thus we have a reason for the use of the arithmetical complement. And if we substitute for the log. of the divisor its arithmetical

complement, then adding instead of subtracting we arrive at a similar result.

$$\begin{aligned} \text{Log. } 418 &= 2\cdot62118 \\ \text{A.C. of } 2\cdot4 &= \bar{1}\cdot61979 \\ \hline \text{Log. } 174\cdot16 &= 2\cdot24097 \end{aligned}$$

This step introduces to our notice the expression $\bar{1}$, or that of a *negative* characteristic; and this again leads us to treat of logarithms of numbers less than unity. The logarithm of a fraction or a number less than unity must possess a negative characteristic; for as the log. of 10 is 1, the log. of $0\cdot1$ is 10^{-1}, and of $0\cdot01$ or $\frac{1}{100}$ or $\frac{1}{10^2}$ is 10^{-2}, according to the rule given on p. 19, which states that a quantity raised to a negative power is equal to the reciprocal of the positive power of that quantity; or $x^{-n} = \frac{1}{x^n}$, whatever number n may stand for. Therefore the exponent or logarithm of a fraction is negative, and for numbers

Between	1	and	$0\cdot1$	lies between	0	and	-1
„	$0\cdot1$	„	$0\cdot01$	„	-1	„	-2
„	$0\cdot01$	„	$0\cdot001$	„	-2	„	-3

Hence the following rule: If the decimal be in the first, second, or third, or nth place to the right, the characteristic will be $\bar{1}$, $\bar{2}$, $\bar{3}$, or $\bar{n}$. Thus in the tables the mantissa of the log. of 275 is $2\cdot43933$; and

$$\begin{aligned} \text{log. } 275 &= 2\cdot43933 \\ \text{log. } 27\cdot5 &= 1\cdot43933 \\ \text{log. } 2\cdot75 &= 0\cdot43933 \\ \text{log. } 0\cdot275 &= \bar{1}\cdot43933 \\ \text{log. } 0\cdot0275 &= \bar{2}\cdot43933 \\ \text{log. } 0\cdot00275 &= \bar{3}\cdot43933 \end{aligned}$$

From this we can clearly understand that the multiplication of a number by any power of 10 affects the characteristic only of the logarithm, and not the mantissa. And from what has gone before, it will appear that given a table of the logarithms of prime numbers* it would be easy to calculate all the logarithms of composite numbers.

The calculations with negative indices are made by the rules of algebra. Thus if it were required to add two of the logarithms given above,—as $2 \cdot 43933$ and $\bar{3} \cdot 43933$,—we add first the mantissæ, and obtain $0 \cdot 87866$, and then prefix the result of adding 2 to $\bar{3}$, viz. $\bar{1}$; $\bar{1} \cdot 87866$ is the log. of $0 \cdot 75625$. As examples of multiplying and dividing with negative characteristics, we may select the following:

To multiply $\bar{3} \cdot 43933$ by 5,

$$\begin{array}{r} \bar{3} \cdot 43933 \\ 5 \\ \hline \overline{13} \cdot 19665 \end{array}$$

The 2 carried on from the last multiplication of the mantissa is added to the $- 15$, and the result $(- 15 + 2) = - 13$, the logarithm of a very small fraction.

To divide $\overline{13} \cdot 19665$ by 5,

The most ready method of division will be to increase the negative characteristic so that it may be an exact multiple of 5, and compensating for the increase, thus:

$\overline{13} \cdot 19665 = \overline{15} + 2 \cdot 19665$, and this divided by 5 gives $\bar{3} + 0 \cdot 43933$ or $\bar{3} \cdot 43933$.

We may add to these examples illustrations of Rules III. and IV. as to the process of involution and evolu-

* A prime or prime number is a number that cannot be divided (except by itself or by unity) without a remainder: 5, 13, 17, 19, 61, etc., are primes.

tion, or the finding of powers and roots by means of logarithms.

Example to Rule III.—Required the sixth power of 18.

$$\begin{array}{r} \text{Log. } 48 = 1\cdot25527 \\ 6 \\ \hline 7\cdot53162 = \text{log. } 34{,}012{,}224. \end{array}$$

Required the sixth root of 34,012,224.

$$\begin{array}{l} 6)\underline{7\cdot53162} = \text{log. } 34{,}012{,}224 \\ \quad 1\cdot25527 = \text{log. } 18. \end{array}$$

Next we shall proceed to the consideration of methods of extending the use of ordinary logarithmic tables, and to explain how a system of logarithms calculated to a base, b, may be recalculated to another base, B.

CHAPTER VII.

TABLES OF LOGARITHMS AND PROPORTIONAL PARTS.

Tables of logarithms are usually arranged to give the mantissæ of the natural numbers consisting of one to five digits,—that is, from 1 to 10,000; and when the given number, or *argument* as it is termed, has no more than five digits, its logarithm is found in the tables on inspection. But it will often occur that the argument has more than five digits, and the logarithm is found by following this reasoning: Let us take the log. of 6276053. Now it can be shown that when any two large numbers differ by unity, the increase of the logarithm is sufficiently nearly proportional for all practical purposes to the increase of the number. We look then in the table for the logarithm

of 62760 and for that of 62761, and find the mantissæ 0·7976829 and 0·7976899 respectively. The logarithm of 6276100 is then = 6·7976899, and that of 6276000 = 6·7976829. Their difference is 0·0000070. Thus for a difference of 100 in the numbers the difference of the logarithms is 0·0000070. We have then the common rule-of-three sum: If 0·0000070 added to the logarithm of 6276000 gives the logarithm of 6276100, what must be added to the log. of 6276000 to give the log. of 6276053?

$$100 : 53 :: 0\cdot0000070 : \text{the number to be added};$$

or

$$\frac{53 \times 0\cdot0000070}{100} = 0\cdot000003710.$$

Disregarding the last two figures, we have

$$\log.\ 6276053 = 6\cdot7976829 + 0\cdot0000037 = 6\cdot7976866.$$

Should the first of the figures omitted, however, be higher than 5, or 5 itself, 1 should be added to the number preceding it.

Had the log. of 627·6053 been required, or that of 0·0006276053, we have seen in the foregoing pages that they would be respectively 2·7976866 and $\bar{4}\cdot7976866$.

Now that we have learned the reason, the rule to find the log. of a given number, whose digits are in excess of those given in the table, may be stated thus: Find and write down from the table the mantissa for as many digits as are given. Take the difference between that mantissa and the next greater. Consider the remaining figures of the given number as a decimal fraction, and multiply the difference by this fraction, rejecting as many figures from the right of the product as the multiplier contains. The result

added to the lesser mantissa is the mantissa of the given number. Study of the example given above will explain this apparently difficult rule.

The number to be added to a lesser log. to give the log. required is termed the *proportional part*, and is given in the best tables. But as these are variously arranged, we leave the student to consult the instructions accompanying the tables he may select, content ourselves with imparting to him the *reason* for what he there will be directed to do.

How to find the number to which a log. corresponds we have next to consider. The number may be found exactly in the table, but it may also occur that the exact number is not given. We then state proportionally the difference between the log. next less and that next greater, and the difference between their corresponding numbers: Thus, suppose we wish to find the number corresponding to the logarithm $3 \cdot 9212074$, the tables give

$$\log. 8340 \cdot 8 = 3 \cdot 9212077$$
$$\log. 8340 \cdot 7 = 3 \cdot 9212025.$$

That is,—if a difference of $0 \cdot 1$ in the numbers gives a difference of $0 \cdot 0000052$ in the logarithms, what difference in the numbers will a difference of $(3 \cdot 9212074 - 3 \cdot 9212025)$ or $0 \cdot 0000049$ in the logarithms give?

The answer is:

$$0 \cdot 0000052 : 0 \cdot 0000049 :: 0 \cdot 1 : 0 \cdot 094,$$

which added to $8340 \cdot 7$ gives $8340 \cdot 794$.

Rule.—Subtract from the given log. the next less tabular log., and this tabular log. from the next greater tabular log. Divide the first difference by the second or tabular difference, and the first two or three figures of the quotient annexed to the figures of the lesser tabular number will be the number required.

Example.

3·9212074	3·9212077
3·9212025	3·9212025
0·0000049	0·0000052
	52)49
	·94

and this annexed to 8340·7 gives 8340·794, as before.

THE ARITHMETICAL COMPLEMENT.—This device, discovered by Gunter about 1614, although of very constant use with logarithms, is not by any means an invariable accompaniment of logarithmic tables. Indeed, the addition of the arithmetical complement to every logarithm would render the tables exceedingly unwieldy, and is the more unnecessary because the complement may be very easily calculated. The rule is—*begin at the left hand, and subtract every figure from* 9 *until the last; subtract that from* 10. The reason for the rule is too obvious to need explanation, but an example may aid the student. Thus, to find the arithmetical complement to the log. 2 = 0·30103,

$$\begin{array}{ccccc} 9 & 9 & 9 & 9 & 10 \\ 3 & 0 & 1 & 0 & 3 \\ \hline 6 & 9 & 8 & 9 & 7 \end{array}$$

and to log. 7 = 0·84510,

$$\begin{array}{ccccc} 9 & 9 & 9 & 10 & \\ 8 & 4 & 5 & 1 & 0 \\ \hline 1 & 5 & 4 & 9 & 0 \end{array}$$

Results which the student can verify by reference to the table on p. 25.

CHAPTER VIII.

TRANSFORMATION OF SYSTEMS OF LOGARITHMS.

HITHERTO we have considered only the common system of logarithms, or those calculated to the base 10. There may be other bases, and consequently other systems of logarithms. The celebrated Baron Napier, the discoverer of logarithms, employed as a base the sum of the following series,* denoting the sum by ϵ.

$$2 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4} \text{ to infinity.}$$

The first eight digits of this sum are 2·7182818, and this is the base of what is known as the Napierian or hyperbolic system of logarithms. This system is used in scientific investigation, and the general principles of the conversion of this system into the common, or of the common into this, will be of service to the student.

Suppose there are any two systems of logarithms whose bases are b and B, and that there is a number n which has x for its log. in the first system and y for its log. in the other system. Then n will be equal to either b^x or B^y, and as we have the first of mathematical authorities—Euclid—for the assertion that "things equal to the same thing are equal to one another," we may say that b^x is equal to B^y. And if we refer to the principle of the equation given in an earlier page we shall see that (which is or should be self-evident) if we divide the x on one side by the y on the other we still maintain an equation, and that $B = b^{\frac{x}{y}}$ (or dividing by y on both sides we

* For an explanation of the term "series" the student is referred to a subsequent chapter.

have $B^{\frac{y}{y}} = b^{\frac{x}{y}}$, the same thing). Therefore (see p. 23) $\frac{x}{y}$ is the log. of B to the base b; and writing this,

$$\frac{x}{y} = \log._b B,$$

we shall perceive the inverse or reciprocal to be

$$\frac{y}{x} = \frac{1}{\log._b B}.$$

The reciprocal is taken to allow of readily multiplying both sides of the equation by x, whence we obtain the value of y, which is

$$y = \frac{1}{\log._b B} x.$$

And this teaches us that if the log. of any number in the system of which the base is b be multiplied by

$$\frac{1}{\log._b B},$$

we shall arrive at the log. of that number to the base B. The multiplier or constant

$$\frac{1}{\log._b B}$$

is termed the *modulus* of the base B with regard to the base b. The modulus of the common system of logarithms, with relation to the Napierian system, is, it follows,

$$\frac{1}{\log._\epsilon 10};$$

and as the numerical value of $\log._\epsilon 10$ is by calculation 2·30258509, this number divided into unity gives 0·43429448, the modulus of the common system, or the number by which the Napierian logarithm must be multi-

plied to give the corresponding logarithm of the common system. Similarly 2·3 etc. is the modulus by which a common logarithm is to be multiplied to obtain the corresponding Napierian logarithm.

In practice the reduction of common to hyperbolic logarithms may be effected, but only approximately, by multiplying the common logarithm by 2·3.

When presently more advanced, the student may with advantage return to this chapter and trace the reasoning embodied in the following formulæ:

It may be shown that

$$\log._b \mathrm{B} \times \log._{\mathrm{B}} b = 1.$$

For if $b^x = \mathrm{B}^y$, then $b = \mathrm{B}^{\frac{y}{x}}$,

$$\mathrm{B} = b^{\frac{x}{y}};$$

and

$$\frac{y}{x} = \log._{\mathrm{B}} b,$$

$$\frac{x}{y} = \log._b \mathrm{B}.$$

Therefore

$$\log._b \mathrm{B} \times \log._{\mathrm{B}} b = \frac{y}{x} \times \frac{x}{y} = 1.$$

Another exercise, and one by which speed and accuracy in calculation may be gained, is the finding of certain logarithms from other given logarithms.

Given the log. of 2 = 0·3010300, to find the logs. of 64 and 128.

Now 64 is the sixth power of 2; hence by Rule III. the log. of 64 will be six times the log. of 2. The log. of 64 is 1·8061800.

The log. of 128 will be the log. of 64 plus the log. of 2, or seven times the log. of 2.

The following example affords very useful illustration :

Given log. 5 = 0·6989700, find the log. of $\sqrt[7]{(6 \cdot 25)}$.

By Rule IV. the log. of the seventh root of 6·25 is equal to the seventh part of the log. of 6·25. Now we may write 6·25 in the form of an improper fraction, as $\frac{625}{100}$, and the resolution of this fraction by logs. according to Rule II., is log. 625 − log. 100, because the form of the fraction indicates that 625 is to be divided by 100; 625, again, is the fourth power of 5, and 2 is the log. of 100. Then

$$(\text{log. } 625 - \text{log. } 100) = (\text{log. } 5^4) - 2.$$

And as the log. of the fourth power of 5 is equal to four times the log. of 5, the expression

$$\frac{(4 \text{ log. } 5) - 2}{7},$$

which may be read four times the log. of 5 less 2 (from the product) divided by 7, will give the answer.

$$\begin{array}{rr}
\text{Log. } 5 = & 0 \cdot 6989700 \\
\text{multiplied by} & 4 \\
\hline
 & 2 \cdot 7958800 \\
\text{less } 2 & 2 \cdot 0000000 \\
\hline
\text{divided by } 7) & 0 \cdot 7958800 \\
\hline
 & 0 \cdot 1136114
\end{array}$$

And by similar applications of the four primary rules many examples may be put and solved by the student himself.

CHAPTER IX.

COMMON USES OF COMMON LOGARITHMS.

There are, of course, many important applications of logarithms that could not be included here. Some of these applications require a technical knowledge of the science to which they are applied; others, again, call for a knowledge of the higher mathematics to which we have not yet attained. But the property of logarithms is best illustrated to the general reader by the consideration of compound interest. On this point no one is so clear as the late Professor De Morgan; and, not to make a vain attempt to embody his meaning in words as concise, it will be better to quote what he says. The quotation is from the 'Essay on Probabilities'* (p. 13).

The principle upon which mathematical abbreviation frequently proceeds is this: that where the calculation of a few results materially aids the production of a great many more, it is advisable to calculate a multitude of results, to arrange them in convenient tables of reference, and to publish them; so that by means of one person taking a little more trouble than would otherwise fall to his share, all others may be saved labour altogether. Mathematical tables are frequently nothing but the result of labour performed once for all; but it also sometimes happens that the principle on which the labour is performed can be exemplified by a familiar case of it. We shall take that of logarithms as an instance.

* 'An Essay on Probabilities, and on their Application to Life Contingencies and Insurance Offices.' By Augustus De Morgan, Professor of Mathematics in University College, London. Longmans and Co. 1838.

Every table of logarithms is an extensive table of compound interest. Not to embarrass ourselves with fractions, let us take a table of *cent. per cent.* compound interest. We have then the following amounts of 1*l.* in 1, 2, 3, etc., years.

Years.	Amount.	Years.	Amount.
	£		£
0	1	14	16,384
1	2	15	32,768
2	4	16	65,536
3	8	17	131,072
4	16	18	262,144
5	32	19	524,288
6	64	20	1,048,576
7	128	21	2,097,152
8	256	22	4,194,304
9	512	23	8,388,608
10	1024	24	16,777,216
11	2048	25	33,554,432
12	4096	26	67,108,864
13	8192	27	134,217,728

The property of this table is, that if we wish to multiply together any two numbers called amounts, we have only to add together the number of years they belong to, and look opposite the sum in the table of years. Thus, 11 and 12 *added* together give 23; 2048 and 4096 *multiplied* together give 8,388,608. The reason is as follows: If 1*l.* in 11 years yield 2048*l.*, and if this 2048*l.* be put out for 12 years more, then, since 1*l.* in 12 years yields 4096*l.*, 2048 times as much will yield 2048×4096; or the amount in $11 + 12$ years is the product of the amounts in 11 and 12 years. The only reason why the preceding table is not in the common sense of the word a table of logarithms is, that its construction leaves out most of the numbers. We can deal with 2048 and 4096, but

there is nothing between them. The remedy is to construct a table of compound interest, at such an excessively small interest, that a year shall never add so much as a pound throughout. Certain considerations, by which the table may be shortened, but with which we have here nothing to do, make it convenient to suppose such a rate of interest, that 1*l.* shall increase to 10*l.* in not less than 100,000 years, at compound interest. Or we may suppose interest payable 100,000 times a year, and say, let the whole *yearly* interest be 1000 per cent. per annum. Taking the first supposition, we have a part of a table of logarithms as follows:

Amount.	Years.	Amount.	Years.
£		£	
1000	300,043	5232	371,867
1001	300,087	5233	371,875
1002	300,130	5234	371,883
1003	300,173	5235	371,892
etc.	etc.	etc.	etc.

This is the light in which a common reader may view a table of logarithms. Let 1 increase to 10 at a compound interest in 100,000 equal moments, then 1 will become 5234 in 371,883 such moments; and so on. We can thus manage to put down every number, within certain limits, as an amount; and thus, within those limits, we reduce all questions of multiplication and division to addition and subtraction by reference to the tables. We thus perceive a simple principle applied with much labour, but such as is performed once for all. The notion above elucidated was the first on which logarithms were constructed; in time came more easy methods.

Professor De Morgan then recurs to another method of

abbreviation, and we may, for the method will be very useful, continue the quotation. The abbreviation is applied to the multiplication of all the successive numbers from 1 up to some higher number.

Let [10], for instance, represent the product of all the numbers, from 1 up to 10, both inclusive, or let [10] stand for $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 = 3{,}628{,}800$.

[1] is 1; [2] is 2; [3] is 6; [4] is 24; [5] is 120; [6] is 720; [7] is 5040; [8] 40,320; and so on. This labour becomes absolutely unbearable when the numbers become larger; thus [30] (or the numbers up to 30 multiplied together) contains 33 places of figures, and [1000] contains 2568 figures. But, nevertheless, we cannot deal with problems in which there are 1000 possible cases without knowing, *either nearly or exactly*, the value of [1000]. It will, however, be sufficient to know this value very nearly, within, say, a thousandth part of the whole; that is, as nearly as when the answer of a problem being 1000, we find something between 999 and 1001. We now put before the reader who can use logarithms a rule for this approximation, with an example; intending thereby to show the reader who does not comprehend the process, how mathematics enter into the abbreviation of tedious computations.

Rule.—To find very nearly the value of [a given number], from the logarithm of that number subtract 0·4342945,* and multiply the difference by the given number, for a first step. Again, to the logarithm of the given number add 0·7981799 and take half the sum, for a second step. Add together the results of the first and second steps, and the sum is nearly the logarithm of the

* These numbers are obtained by calculation, which need not be given here.

product of all numbers up to the given number inclusive. For still greater exactness, add to the final result its aliquot part, whose divisor is 12 times the given number.

Example.—What is

$$[30] \text{ or } 1 \times 2 \times 3 \times 4 \times \ldots\ldots \times 29 \times 30\,?$$

First step.

$$\begin{array}{rr} \log.\ 30 = & 1\cdot4771213 \\ & 0\cdot4342945 \\ \hline & 1\cdot0428268 \\ & 30 \\ \hline & 31\cdot284804 \text{ Result.} \end{array}$$

Second step.

$$\begin{array}{rr} \log.\ 30 = & 1\cdot4771213 \\ & 0\cdot7981799 \\ \hline & 2)2\cdot2753012 \\ \hline & 1\cdot1376506 \text{ Result.} \end{array}$$

The sum of the results is

$$32\cdot422455,$$

which is the log. of result of problem.

The result has therefore 33 places of figures, of which the first six are (nearly) 264,518; or, if this be increased by its 360th (12 times 30) part, or about 735, the result is 265,253, followed by 27 ciphers, and the error is not so much of the whole as one part out of 500,000. In this way, we are able to do with more than sufficient nearness, and in a few minutes, what it would take days to arrive at

by the common method and with much greater risk of error.

If we wish to find the product of all the numbers, say, from 31 to 100, both inclusive, we find [100] and [30] approximately, and divide the first by the second.

This selection from a work that should be better known to the ordinary reader who has a mathematical tendency, will serve to illustrate the application of logarithms to extremely lengthy calculations. Another application now to be considered is to the immediate computation of amounts at compound interest, a class of calculation in which logarithms are of high value.

Before the subject can come within easy grasp, there will be necessary some explanation of other terms than those already made known. An amount at compound interest increases in what is named *geometrical progression.*

Arithmetical and Geometrical Progression.

An amount (a) is ordinarily understood to be capable of increase by one of two methods:

(1). It may increase by a constant *difference* d, as

$$a,\ a+d,\ a+2d,\ a+3d, \text{ etc.},$$

when the progression is said to be *arithmetical.*

For example,

$$1, 2, 3, \text{ etc.},$$

is an arithmetical progression, the difference d being 1. This progression might be written

$$1,\ 1+1,\ 1+1+1, \text{ etc.}$$

It will be seen on trial that the last term (l) can be obtained from the formula

$$l = a + (n - 1)\,d,$$

where n is the number of terms, a the first term, and d the common difference. Thus—to give a self-evident example—the last term of the arithmetical series,

$$1, 2, 3, 4, 5,$$

is

$$1 + (5 - 1)\,1 = 5.$$

(2). Or an amount may increase by multiplication by a constant quantity. An amount is then said to increase in *geometrical progression*, as

$$2, 4, 8, 16, 32, \text{etc.}$$

While, in any *arithmetical* series or progression, a number or term subtracted from the next following number or term, yields a constant *difference;* in a *geometrical series*, a term divided into the next term gives a constant *quotient or ratio.* The common ratio of the series just given is 2. The general algebraic formula for the geometrical series is

$$a,\ a\,r,\ (a\,r)\,r,\ (a\,r\,r), \text{etc.,}$$

or

$$a,\ a\,r,\ a\,r^2,\ a\,r^3, \text{etc.,}$$

a being the first term, and r the common ratio or constant multiplier. Study of the series will show us that the last term (l) will be equal to

$$l = a\,r^{n-1}.$$

The last term in the given series is thus:

$$l = 2\,(2^{5-1}) = 2\,(2^4) = 2 \times 16 = 32.$$

An amount increasing at compound interest or in geometrical progression, is now put before us in a new light; for, if it be required to know—what will be the amount of a given sum placed at compound interest for so many years, at such a rate per cent.—we can put the question under the form—what is the last term (the amount) of a geometrical series, to be obtained from a knowledge of the first term (the sum placed out), the number of terms (years), and the common ratio (rate per cent.).

Example.—Required the amount, at compound interest, of 2273*l.*, in 3 years, at 5 per cent.?

Here 2273 is the first term; 3 the number of terms; and $1 \cdot 05$ (the amount of 1*l.* for 1 year) the common ratio. Now the formula $l = ar^n$ directs us to raise r, the common ratio, to the nth power, and to multiply the result by the first term. The log. of $r = 1 \cdot 05$ is $0 \cdot 0211893$, and this multiplied by n or 3 (see Rule III. for logarithms) is $= 0 \cdot 0635679$.

$$\begin{aligned} \log.\ (1 \cdot 05 \times 3) &= 0 \cdot 0635679 \\ \log.\ \quad 2273 &= 3 \cdot 3565994 \\ \hline 3 \cdot 4201673 &= \log.\ 2631 \cdot 2. \end{aligned}$$

Answer, 2631*l.* 10*s.*

Had we from the amount 2631*l.* 10*s.* sought the principal, the interest being given, the following process—the reverse of the preceding—would have been adopted:

$$2631 \cdot 2 \div 1 \cdot 05^3 = \text{principal, or}$$

$$\frac{l}{r^n} = a.$$

$$\begin{aligned} \log.\ 2631\cdot 2 &= 3\cdot 4201673 \\ \log.\ \ 1\cdot 05^3 &= 0\cdot 0635679 \\ \hline & 3\cdot 3565994 = \log.\ 2273. \end{aligned}$$

Had the rate of interest been demanded, the method employed would obviously have been that indicated by the expression

$$1\cdot 05^3 = 2631\cdot 2 \div 2273,\text{ since}$$

$$2273 \times 1\cdot 05^3 = 2631\cdot 2\ ;\ \text{or } r^n = \frac{l}{a}\cdot$$

$$\begin{aligned} \log.\ 2631\cdot 2 &= 3\cdot 4201673 \\ \log.\ 2273 \quad &= 3\cdot 3565994 \\ \hline & 3)0\cdot 0635679 \\ \hline & 0\cdot 0211893 = \log.\ 1\cdot 05. \end{aligned}$$

$1\cdot 05$ is an increment to one of $0\cdot 05$ or 5 in the hundred (per centum).

Had the time been required, we find

$$\log.\ 1\cdot 05^3 \div \log.\ 1\cdot 05 = 3,\text{ the number of years;}$$

or

$$\frac{r^n}{r} = n.$$

The student will now be able to take up with clearer understanding the rules given in those treatises dealing specially with logarithms; it is necessary to resume our consideration of algebraic principles.

CHAPTER X.

COMPOUND MULTIPLICATION AND THE BINOMIAL THEOREM.

The reader will have seen that the repeated addition of a quantity to itself is termed multiplication. To illustrate this familiarly, let us understand by the letter a the possession of one apple. Then it will be perceived that multiplication by 1 or $1a$, or once a, is equal to a, showing that the coefficient 1 need not be written. Indeed, 1 as a coefficient is always omitted. But multiplying a by 2 is to take two apples, $a + a$; and so on. In cases of numerical coefficients of higher value than 1 or unity, the coefficient cannot be omitted.

To multiply by a fraction, it must follow, is to take a certain portion of the quantity multiplied. The multiplication of a by $\frac{1}{2}$ is represented by taking half an apple; the multiplication of $4a$ by $\frac{1}{2}$ by taking 2 apples from 4.

Whilst the multiplication of a whole quantity, therefore, by a quantity greater than unity, results in a product *greater* than the quantity multiplied; the multiplication of a whole quantity by a fractional quantity results in a product *less* than the quantity multiplied. For the want of this, or similar axiomatic reasoning, the student is often, for a moment, confused, when he finds that the result of *multiplication* is a *lesser* quantity than the quantity multiplied.

He will also see by trial that as the value of the multiplier is gradually reduced, so is reduced the value of the product; and when the value of the multiplier equals 0, the product will be 0. Thus $a \times 0 = 0$; and equally $a \times b \times c \times 0 = 0$; so that where 0 enters as a factor, the product is 0.

To multiply together fractions, is to take a part or

parts of the part or parts of a quantity. To multiply $\frac{1}{2}a$ by $\frac{1}{4}$ is represented by taking one-fourth of half an apple, giving the product of one-eighth of an apple or $\frac{1}{8}a$.

Had it been required to multiply $\frac{1}{2}a$ by $\frac{2}{3}$, the process would be represented by taking half an apple, dividing this half into three parts, and taking two of these parts, or two-sixths of the entire apple, $\frac{2}{6}a$. This reasoning, which it will be necessary to resume in the consideration of fractional quantities, is too closely consequent upon the previous study of multiplication to be here omitted; and the reader who wishes a more extended practice in mathematics than these present pages are intended to convey, will do well to make himself familiar with the instances given by substituting other quantities.

Although it is not productive of difference in the results of multiplication, the arrangement of literal coefficients generally follows alphabetical order. Let $a = 4$ and $b = 5$; the result of the operation indicated by $a \times b$, $a.b$ or ab—and by $b \times a$, $b.a$ or ba—namely, that of multiplying in the one case a by b, and in the other b by a—is the same.

The multiplication of compound quantities is perhaps best considered from the view-point of multiplying, say $4a$ by $5b$. The operation indicated is that of taking $4a$, $5b$ times, or $5b$, $4a$ times. If we commence by taking $5b$ a times, we have the result $5ab$. This result is to be taken 4 times to give the final result of $20ab$.

As has been stated (p. 14), these rules hold in all our systems of mathematics, except in "Quaternions" and "Double Algebra."

Thus we have the general rules:

1. That when factors are multiplied, the order of their multiplication is without effect upon the product.

2. That if the factors are compound quantities, each term of one factor must be multiplied into each term of the other factor.

A compound quantity is, however, generally understood to consist of two or more quantities connected by the signs (+) plus or minus (−); $a + x$ is thus a compound quantity, and $a x$ a simple quantity, because in the latter case the signs of + and − do not exist.

The members that compose a compound quantity (for instance the a or the x in the compound expression $a + x$) are *called* terms; and a quantity of

Two terms is known as a binomial,			
Three	„	„	trinomial,
Four	„	„	quadrinomial, etc.

A quantity of more than one term is known generally as a polynomial—so that a polynomial may contain two, three, four, etc., or any number of terms greater than one term.

The distinction between simple and compound quantities is one of great importance, and is founded on the following principle. Let it be required to multiply $a \times b \times c \times x$ by n; the product is

$$a \times b \times c \times x \times n;$$

or

$$a b c n x.$$

But if the quantity to be multiplied by n were $a + b + c + x$, the product would be

$$n a + n b + n c + n x.$$

Hence rule 2. In the first instance we multiply a product by n; in the second we multiply each of the terms of

a sum. If figures are substituted for letters, the distinction is at once evident: thus in the first instance,

$$\begin{array}{ccccccccc} a & \times & b & \times & c & \times & x & \times & n \\ \parallel & & \parallel & & \parallel & & \parallel & & \parallel \\ 1 & \times & 2 & \times & 3 & \times & 4 & \times & 5 = 120. \end{array}$$

In the second,

$$\begin{array}{ccccccc} na & + & nb & + & nc & + & nx \\ \parallel & & \parallel & & \parallel & & \parallel \\ 1 \times 5 & + & 2 \times 5 & + & 3 \times 5 & + & 4 \times 5 = 50. \end{array}$$

In the second instance the quantity $a + b + c + x$ is compound, and the quantity n simple. The next step to consider will be the multiplication of factors, both compound. For example, let it be required to multiply $a + b$ by $c + x$. The operation to be performed is to multiply $c + x$ by a and by b, and to take the sum of the products. $c + x$ multiplied by a gives $ac + ax$; and $c + x$ multiplied by b gives $bc + bx$; adding we have the final product $ac + ax + bc + bx$.

Before we proceed to Newton's great discovery—the binomial theorem—it will aid us to recur to the principle that regulates the signs of the quantities of the product. Multiplication considered as the process of repeated addition affords easy verification of the rule that the multiplication of quantities with like signs gives plus, and with unlike signs minus, quantities. Thus x or $+x$ multiplied by -3 is represented on the principle of addition by $+x + x + x$, which is to be *subtracted* from the other quantities (if any) with which the quantity x may be allied, and is consequently $-3x$. That is, we are to subtract three times x. The distinction, then, between multi-

plying by a positive and negative factor, is that the sum of the repeated addition is, in the case of positive signs, to be added to the other terms (if any); or in the case of negative signs, to be subtracted from the other terms (if any). If there are no other terms, the product stands alone; in the case of the product of unlike signs, the negative sign is prefixed.

To illustrate this, let us multiply $a + b$ by $a + b$; $a + b$ by $a - b$; and $a - b$ by $a - b$.

$$\left.\begin{array}{l} a + b \\ a + b \end{array}\right\} \text{ or } (a + b)^2$$

$$\begin{array}{llll} aa & + ab & & \\ & + ab & + bb & \\ \hline aa & + 2ab & + bb & = a^2 + 2ab + b^2. \end{array}$$

From this we learn that $a + b$ multiplied into itself, or squared, gives as product the sum of the squares of the two terms *plus* twice their product.

$$\left.\begin{array}{l} a - b \\ a + b \end{array}\right\} \text{ or } (a - b)\ (a + b)$$

$$\begin{array}{llll} aa & - ab & & \\ & + ab & - bb & \\ \hline aa & & - bb & = a^2 - b^2. \end{array}$$

$$\left.\begin{array}{l} a - b \\ a - b \end{array}\right\} \text{ or } (a - b)^2$$

$$\begin{array}{llll} aa & - ab & & \\ & - ab & + bb & \\ \hline aa & - 2ab & + bb & = a^2 + b^2 - 2ab. \end{array}$$

And from the last calculation we see that the square of the difference of a and b (and what is true of a and b should be true of all quantities) is equal to the sum of the squares of the two terms *less* twice their product.

We are now in a position to understand the method of the binomial theorem.

The Binomial Theorem.

So important is thought by mathematicians the method of the binomial theorem, that the discovery is recorded upon the tomb of its inventor, Sir Isaac Newton. The reader will perceive the value of the invention when he is told that the binomial theorem admits of the immediate involution of binomial quantities to any proposed power. The involution of $(a + b)$ to so low a power as the fourth entails by the ordinary process of multiplication considerable labour; but to involve a high power would need days or even weeks of labour, liable at all times to error. By the binomial theorem the involution may be effected almost at sight.

Let us commence investigation of the method by involving $a + b$ to the second, third, and fourth powers.

$$(a + b)^2 = a^2 + 2\,a\,b + b^2.$$
$$(a + b)^3 = a^3 + 3\,a^2 b + 3\,a\,b^2 + b^3.$$
$$(a + b)^4 = a^4 + 4\,a^3 b + 6\,a^2 b^2 + 4\,a\,b^3 + b^4.$$

Also

$$(a - b)^2 = a^2 - 2\,a\,b + b^2.$$
$$(a - b)^3 = a^3 - 3\,a^2 b + 3\,a\,b^2 - b^3.$$
$$(a - b)^4 = a^4 - 4\,a^3 b + 6\,a^2 b^2 - 4\,a\,b^3 + b^4.$$

Study of these tabulated results of involution will lead us to the principles by which we may write down the power of any binomial.

First, we perceive that the number of terms is always one greater than the power required.

Secondly, if both terms of the binomial are positive, all the quantities remain positive; but that a residual* binomial has its odd terms (counting from the left) positive, and its even terms negative.

Third. The exponent of the first or *leading* quantity (a) is always the index of the required power; and the powers of this quantity decrease by one in each successive term. The index of the second term of the quantity to be involved (b) (called the *following* quantity), begins with 1 in the second term, and increases by 1 in each successive term until it equals the required power.

Fourth. The coefficient of the first and last terms is always 1. The coefficient of any term multiplied by the index of the first letter (leading quantity) in that term, and divided by the index of the second letter (following quantity) increased by 1, gives the coefficient of the next succeeding term. The coefficient of the second, and next to the last terms, is the index of the required power.

The algebraic formulation of the theorem is

$$(a+b)^n = a^n + n\,a^{n-1}\,b + n\,.\,\frac{n-1}{2}\,a^{n-2}\,b^2 +$$

$$n\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3}\,a^{n-3}\,b^3 +, \text{ etc.}$$

By the substitution of the letters A, B, C, for the coefficients,

$$n,\ n\,.\,\frac{n-1}{2},\ n\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3},\ \text{etc.,}$$

* A residual quantity expresses the difference of two quantities, as $a - b$.

the formula is much simplified, and becomes

$$(a+b)^n = a^n + \mathrm{A}\,a^{n-1}b + \mathrm{B}\,a^{n-2}b^2 + \mathrm{C}\,a^{n-3}b^3, \text{ etc.}$$

Let us now endeavour, by following the foregoing rules, to raise $a + b$ to the sixth power. By the first principle, the number of terms will be seven. The signs will be plus.

The indices of the seven terms will be a^6, $a^5 b$, $a^4 b^2$, $a^3 b^3$, $a^2 b^4$, $a b^5$, b^6.

The coefficients will be

$$1,\ 6,\ \frac{6\times 5}{2},\ \frac{15\times 4}{2},\ \frac{20\times 3}{4},\ 6,\ 1.$$

Collecting these steps, we have the complete expression

$$a^6 + 6\,a^5 b + 15\,a^4 b^2 + 20\,a^3 b^3 + 15\,a^2 b^4 + 6\,a b^5 + b^6.$$

By substitution, other binomials may be reduced to the form $(a+b)^n$. Thus if the sixth power of $(3\,d + 2\,e)$ were required, the result could be obtained by substituting a for $3\,d$, and b for $2\,e$. Then, raising $a + b$ to the sixth power, as given above, we afterwards obtain, by restoring the values of a and b, the expression

$$729\,d^6 + 2916\,d^5 e + 4860\,d^4 e^2 + 4320\,d^3 e^3 + 2160\,d^2 e^4 + 576\,d e^5 + 64\,e^6.$$

Or it may occur that one of the terms of the binomial is a unit, as $(a+1)^3$. In this case, since every power of 1 is 1, and multiplication by 1 is unproductive of alteration in the product, the unit is usually omitted, except in its essential place, the first or last term.

For example:

$$(a+1)^3 = a^3 + 3\,a^2 + 3\,a + 1;$$

and

$$(1+a)^n = 1 + n\,x + n\,.\,\frac{n-1}{2}\,x^2 +, \text{ etc.}$$

By similar tact in dealing with problems, the binomial theorem may be made to include the involution of quantities of more than two terms. In the event also of one of the terms of a binomial being a fraction, as $(a + \frac{1}{2})^2$, tact affords a far better method than rule in solving the question.

Thus $a + \frac{1}{2}$ may be written $\frac{2a+1}{2}$,

and

$$\left(\frac{2a+1}{2}\right)^2 = \frac{4a^2+4a+1}{4} = a^2 + a + \tfrac{1}{4}.$$

CHAPTER XI.

DIVISION, FRACTIONS, AND RATIO.

The reader may ask why the common order of mathematical books is departed from, by bringing to one focus the consideration of the principles of so apparently diverse subjects as division, fractions, and ratio. Perhaps the most efficient answer will be found in a general statement of the symbols employed in the algebraic representation of these processes.

I. The operation of *division* is, algebraically, represented in two ways:

a. The quantity to be divided (the *dividend*) is written before the sign $\div$ (to be read "divided by"), and this sign precedes the dividing quantity (the *divisor*). Thus $a \div b$ is to be understood to indicate the division of a by b.

b. The dividend is written above a short line, the divisor below the line, thus, $\frac{a}{b}$.

In both cases the result of the division is termed the *quotient.* And the entire operation and result may be stated in the form of an equation, either as

$$(a).\ a \div b = q,$$

or as

$$(b).\ \frac{a}{b} = q.$$

II. The last expression, $\frac{a}{b}$, is the general form by which is represented a *fraction;* illustrated by supposing a to represent an apple, and b the number of parts into which it is to be divided. If we divide one apple into two parts, we have numerically $\frac{1}{2}$, or one-half as the result. And generally if we divide n things into p portions, or among p persons, each portion will be, or each person will get that quantity of the things represented by $\frac{n}{p}$.

III. There are two methods of representing the relation or ratio of two quantities:

(a). The most useful is in the form of a fraction, as $\frac{n}{p}$; here the ratio expressed is that of the number of things to the number of persons.

(b). The other method of representing the ratio two quantities bear to each other is by placing between them a colon (:),—thus $n : p$ has the same meaning as $\frac{n}{p}$, both expressions indicating the relation of the quantities.

Proportion is the equality of ratios. It is a common error to say that "two numbers bear a certain *proportion*"; correct mathematical language would dictate "two

numbers bear a certain *ratio*." Only *two ratios* can be said to be *proportionate*. Thus, if we have to divide a apples amongst b boys, or algebraically $\frac{a}{b}$; and if we have to divide n things amongst p persons, or algebraically $\frac{n}{p}$; then if the two ratios $\frac{a}{b}$ (ratio of the number of apples to the number of boys) and $\frac{n}{p}$ (ratio of number of things to number of persons) be equal, we may place between them the sign of equality (=), and the equation expresses a proportion,

$$\frac{a}{b}=\frac{n}{p}$$

or

$$a : b = n : p$$

or, again, substituting the sign : : for the sign of equality,

$$a : b :: n : p$$

This we see is a return to an earlier arithmetical stage of a rule-of-three sum. For suppose 6 apples amongst 3 boys and 4 things amongst 2 persons, we have

$$6 : 3 :: 4 : 2$$

The relation, therefore, of division, fractions, and ratio, is sufficiently close to call for their consideration under one head. The expression

$$\frac{a}{b}$$

may, we have seen, mean,

I. The number of times a contains b.

II. A fraction, a being the *numerator* and b the *denominator.*

III. The relation or ratio of a and b.

These three heads really express, under different forms, the same notion—the relation of a to b. In each case we may determine the value of the relation by a process of continued subtraction (as in multiplication we arrived at the product by a process of continued addition). Suppose, as a first example, that a represents 10, and the numerical value of b is 2; we shall find the value of the relation of a to b or $\frac{a}{b}$ to be 5; for we can subtract 2 from 10 five times.*

10	
2	1st subtraction
—	
8	
2	2nd "
—	
6	
2	3rd "
—	
4	
2	4th "
—	
2	
2	5th "
—	
0	

* Had 11 been taken for a, there would have been a remainder $(r) = 1$, and the general algebraic expression would become (where q represents the quotient)

$$\frac{a}{b} = q + r.$$

Suppose, as a second instance, that a should represent 2 and b equal 10; then we should know that we have to divide 2 into 10 parts or 1 into 5 parts, taking one of those parts, or $\frac{1}{5}$th.

Let us now study the effect of different algebraic arrangements of the general expression

$$\frac{a}{b} = q, \text{ or } \frac{10}{2} = 5.$$

In words, this tells us that the result of dividing the dividend (a) by the divisor (b) is the quotient (q).

Again:

$$a = q\,b, \text{ or } 10 = 5 \times 2,$$

or the dividend equals the quotient multiplied by the divisor, and

$$b = \frac{a}{q}, \text{ or } 2 = \frac{10}{5},$$

or the divisor equals the dividend divided by the quotient.

Besides these equations, there are others expressing fundamental principles, and the chief of these is

$$\frac{n\,a}{n\,b} = \frac{a}{b};$$

or the multiplication of the numerator and denominator of a fraction, or the multiplication of the dividend and divisor, by an equal quantity, does not affect the value of the fraction nor of the quotient. Whence it may be perceived:

That the multiplication of the numerator of a fraction by an integer increases the value of the fraction.

That the multiplication of a dividend by an integral quantity increases the value of the quotient.

That conversely the multiplication of a denominator by an integral quantity decreases the value of a fraction.

That the multiplication of a divisor by an integral quantity decreases the value of the quotient.

If we substitute for the term "integral quantity" or "integer" (that is, *whole* number) a fractional quantity, of course the converse only of each rule will hold good.

Rules for Division.

If we observe the rule (the same as for multiplication) that *division of* LIKE *signs yields a* POSITIVE *quotient, of* UNLIKE *signs a* NEGATIVE *quotient,* we may reduce the numerous rules for division given in many of our text-books to the following:

(1). To divide one *monomial* by another,—suppress the letters that are common to both; subtract the exponents which affect the same letters; and divide the coefficients one by another.

(2). To divide a *polynomial* by a *monomial,*—divide each term of the polynomial by the monomial according to the first rule; connect the results by their proper signs (remembering like produce positive, unlike produce negative, signs).

(3). To divide two polynomials one by the other,—arrange them with respect to the same letter; then divide the first terms one by the other, and thence will result one term of the quotient; multiply the divisor by this, and subtract the product from the dividend: proceed to the end in the same manner.

(4). To divide algebraic fractions,—invert the terms of

the divisor, and multiply together the numerators and the denominators.

Examples to these Rules:

(1). $\dfrac{12\,a^3b^2c}{3\,a\,b} = \dfrac{12}{3}a^{3-1}\,b^{2-1}\,c = 4\,a^2\,b\,c.$

(2). $\dfrac{6a^2+12ab-9abc}{3a} = \dfrac{6a^2}{3a} + \dfrac{12ab}{3a} - \dfrac{9abc}{3a} = 2a+4b-3bc.$

(3). $\dfrac{a^3-3\,a^2b+3\,ab^2-b^3}{a-b}.$

$$
\begin{array}{lll}
\text{Divisor.} & \text{Dividend.} & \text{Quotient.} \\
a-b) & a^3-3\,a^2b+3\,ab^2-b^3 & (a^2-2\,a\,b+b^2 \\
 & a^3-\;\;a^2b & \\
 & \overline{\quad -2\,a^2b+3\,ab^2} & \\
 & \quad -2\,a^2b+2\,ab^2 & \\
 & \overline{\qquad\qquad ab^2-b^3} & \\
 & \qquad\qquad ab^2-b^3 & \\
\end{array}
$$

CHAPTER XII.

DIVISION, FRACTIONS, AND RATIO (*continued*).

We may now turn our attention to fractions. Fractions appear to be to many the most difficult section of elementary mathematics; and treated in the abstract there is no doubt there must be attained a high mental culture before their meaning and powers can be fully understood. Indeed, it would seem impossible to impart to one entirely ignorant of mathematics a knowledge of fractions from

merely their abstract consideration. In dealing with this section of our subject it will, therefore, be forgiven us if we incline to more concrete detail than the reader may think necessary.

What is a fraction? A fraction, says Mr. Merrifield, in his admirable text-book of arithmetic,—a fraction of anything is a given portion of it. If we divide anything, no matter how, any piece of it is a fraction of it. The simplest fraction is *one-half*, and is expressed as follows: $\frac{1}{2}$. If we divide a thing, say a cake, into three equal pieces, and take one of them, we call that piece *one-third* of the cake, and we express it as a fraction thus: $\frac{1}{3}$; and so if we divide it into four, and take one portion, we call it a quarter, or one-fourth, and write it $\frac{1}{4}$. So, also, if we divide a cake into x pieces, we should call each piece $\frac{1}{x}$; if into a pieces, each would be $\frac{1}{a}$. It is evident that the whole cake may be considered as two halves (which we write $\frac{2}{2}$), three thirds ($\frac{3}{3}$), four quarters ($\frac{4}{4}$), and the same with any other number. Again, since we cannot divide a cake into less than two pieces, it is clear that division by the number *one* leaves it just as it was, one whole cake, which we may therefore write $\frac{1}{1}$, and this is all we mean by writing

$$1 = \tfrac{1}{1} = \tfrac{2}{2} = \tfrac{3}{3} = \tfrac{4}{4} = \tfrac{1001}{1001}, \text{ etc.}$$

If we divide the cake into a good many parts (say 12), and take any number of these parts (say 7), we express the whole portion taken thus: $\frac{7}{12}$. The number below the line is called the denominator, and expresses the number of parts into which we divide the thing; the

number above the line is called the numerator, and expresses the number of parts taken:

$$\text{Fraction} = \frac{\text{numerator}}{\text{denominator}}$$

Premising that our arithmetical knowledge extends thus far, we may take those rules given for the working of fractions, and endeavour to see in what manner they apply. The rules are concisely stated, as

Rules for Fractions.

Addition and Subtraction.—If the fractions have a common denominator, add or subtract the numerators, and place the sum or difference as a new numerator over the common denominator.

If the fractions have not a common denominator, they must be reduced to that state before the operation is performed.

Multiplication and Division.—To multiply a fraction by an integral quantity, multiply the numerator by that quantity, and retain the denominator.

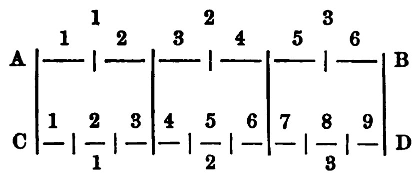
To divide a fraction by an integral quantity, multiply the denominator by that number, and retain the numerator.

To multiply two or more fractions is the same as to take a fraction of a fraction; and is, therefore, effected by taking the product of the enumerators for a new numerator, and of the denominators for a new denominator. (The product is evidently smaller than either factor when each is less than unity.)

To divide one fraction by another, invert the divisor and proceed as in multiplication. (The quotient is always

greater than the dividend when the divisor is less than unity.)

Now we find in the first words of our first rule the phrase "common denominator." We may to the better understanding of this phrase, take an arithmetical example. Let us divide two lines into a number of equal parts—the line A B into 6 parts, and the line C D into 9 parts:



Now the line A B will represent a denominator of 6, and the line C D a denominator of 9. For a numerator (the number of parts we take) let us write down 2. We see that two-sixths, three-ninths, and one-third are the same. So that were we asked to add together three-ninths and one-third, we should reply that the answer was two-thirds. Unconsciously we have performed the operation known as "*reducing the fractions* ($\frac{3}{9}$ and $\frac{1}{3}$) *to a common denominator*"—that is, we have found the value of three-ninths in thirds. While the *denominations* of the quantities were different we could not add them, but directly the value of one in terms of the other was known addition could be proceeded with.

To reduce fractions to a common name or denominator the following rule is generally given:

Multiply *each* numerator by the *product* of all the denominators *except its own;* the results will be the numerators of the new fractions.

The *product* of *all* the denominators will be the new denominator common to all the new fractions.

To apply our rule, let it be required to add

$$\frac{a}{b} + \frac{c}{d} + \frac{e}{f}.$$

We have

$$\frac{adf}{bdf} + \frac{bcf}{bdf} + \frac{bde}{bdf} = \frac{adf + bcf + bde}{bdf}$$

as the sum. The fractions

$$\frac{a}{b} + \frac{c}{d} + \frac{e}{f}$$

have not been altered; they have by the multiplication, in each case of numerator and denominator by an equal quantity (see p. 58), taken a new form. The equal quantities are for a over b, df; for c and d, bf; and for e and f, bd. And these equal quantities have been determined by the fact that bdf is a product that contains all these factors—in other words, by the fact that bdf is a common multiple of b and d and f. The reduction of fractions to a common denominator is then resolved into finding a common multiple of the denominators of all the fractions, and multiplying the numerator of each fraction by the factors foreign to itself.

It may be remarked that if a common multiple have a common measure or common divisor, the result of the division of the multiple by the *greatest common divisor* or *measure* is termed the *least common multiple*. These phrases are sometimes expressed by the initial letters G. C. M. (greatest common measure) and L. G. M. (least common multiple). A common multiple of 6 and 4 is 24;

but the numbers 6 and 4 have a divisor 2 common to both, and the *least* common multiple of 6 and 4 is therefore 12.

The following formulæ will serve to illustrate the principles of the preceding sections, and to render the reader familiar with the expression of principles in symbols:

$$\frac{a}{b} = \frac{m\,a}{m\,b} \qquad \frac{a}{b} = \frac{1}{b} \times a.$$

$$\frac{a}{b} \text{ and } \frac{c}{d} = \frac{a\,d}{b\,d} \text{ and } \frac{b\,c}{b\,d}.$$

$$\frac{a}{b} + \frac{c}{d} = \frac{a\,d + b\,c}{b\,d}.$$

$$\frac{a}{b} - \frac{c}{d} = \frac{a\,d - b\,c}{b\,d}.$$

$$\frac{a}{c} \mp \frac{b}{c} = \frac{a \mp b}{c}.$$

$a \mp b$ should be read, a minus *or* plus b, addition or subtraction being performed as dictated by the application of the formula.

$$\frac{a}{b} \times \frac{c}{d} = \frac{a\,c}{b\,d}$$

$$\frac{a}{b} \text{ divided by } \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{a\,d}{b\,c};$$

or

$$\frac{\;\frac{a}{b}\;}{\;\frac{c}{d}\;} = \frac{a\,d}{b\,c}.$$

$$\frac{a}{bc} = \frac{a}{b} : c = \frac{a}{c} : b$$

$$\frac{a}{b} = \frac{am}{bm} = \frac{a:m}{b:m}$$

$$\frac{ab}{c} = \frac{a}{c}b = a\frac{b}{c}.$$

And the following example, given by Prof. De Morgan, will prove useful:

$$\frac{1}{a+\frac{1}{b}} = \frac{b}{ab+1}.$$

$$\frac{1}{a+\frac{1}{b+\frac{1}{c}}} = \frac{1}{a+\frac{c}{bc+1}} = \frac{bc+1}{abc+a+c}.$$

Thus:

$$\frac{1}{6+\frac{1}{7+\frac{1}{8}}} = \frac{1}{6+\frac{8}{57}} = \frac{57}{350}.$$

CHAPTER XIII.

DIVISION, FRACTIONS, AND RATIO (*continued*).

It is generally required, in any question where two or more quantities are concerned, to ascertain the relationship of these quantities. The relationship is henceforth known as the *difference*, or as the *ratio*. If of two sets, each set of two quantities, the differences or ratio are similar, then those quantities are said to be *proportional*, arithmetically or geometrically.

An arithmetical proportion is an equality of *differences*,

or *equi-difference*; a geometrical proportion is an equality of ratios. $a - b = c - d$ is an arithmetical proportion; $a : b = c : d$ is a geometrical proportion.

When four terms or quantities are thus placed, the first and last are called the *extremes*, the second and third the *means*.

In an arithmetical proportion, the *sum* of the extremes is equal to the *sum* of the means; in a geometrical proportion, the *product* of the extremes is equal to the product of the means. Thus, with

$$(1).\ a - b = c - d$$

and

$$(2).\ a : b = c : d$$

$a + d = b + c$ and $a d = b c$ in each case respectively. Thus it follows from (2) that when two fractions are equal, the numerator of one of them is to its denominator, as the numerator of the other is to its denominator. And it also follows that a proportion may always be constituted with the factors of two equal products.

If four terms are proportional, they are proportional in any other order, so that we preserve *similarity** in the order. Thus:

$$\begin{array}{ll}(a). & 5 : 3 :: 15 : 9 \\ (b). & 5 : 15 :: 3 : 9 \\ (c). & 9 : 3 :: 15 : 5 \\ (d). & 9 : 15 :: 3 : 5\end{array}$$

The position of the extremes is here unchanged, while that of the means is varied; this process was termed by the older writers on mathematics *alternando*, by the

* Two extremes or two means are "similar" terms; an extreme and a mean "dissimilar" terms.

moderns *alternation*. Placing the extremes in the places of the means, as

$$3 : 5 :: 9 : 15$$

This was called *invertendo*, and by the moderns is termed the inversion of the terms.

If there be two proportions, as (a) and (b) above, then multiplying them term by term, the products are proportional, as (25 : 45 : : 45 : 81). This is the principle of the arithmetical process known as the "compound rule of three" or "compound proportion."

The process known in arithmetic as "rule of three," or "simple proportion," is derived from the fact that if one of the terms (x) of a proportion (as $a : b :: c : x$) be unknown, it may be ascertained from the law that a multiplied by x will equal b multiplied by c, and that to find the value of x, it is only necessary to divide the product of b and c by a; or $x = \frac{b\,c}{a}$. In other words, we multiply the second and third terms together, and divide by the first, to find the fourth term, or the answer.

In arithmetic, too, we are told that "the mean proportional between two numbers is equal to the square root of their product." The reason we may perceive from the following proportion:

$$a : b :: b : c$$

Here b is what is termed a mean proportional. The product of the means is b^2, and we know that $b^2 = a\,c$, the product of the extremes. To find then the mean proportional to $a\,c$, we take the square root of the product; for if $a\,c = b^2$ then $\sqrt{a\,c} = b$.

Indeed, by deducing different equations from the equal

ratios (or fractions) $\frac{a}{b} = \frac{c}{d}$ we may arrive at endless sets of proportionals. Professor Young says very clearly upon this point:

"If two fractions, $\frac{a}{b} = \frac{c}{d}$ are equal, then we may replace the terms a, b of the first by any expressions involving a and b that are *homogeneous* * in reference to those quantities, provided we also replace the terms c, d by expressions involving c and d in the same manner. For instance: $2\,a^2 - 3\,a\,b + b^2$ is homogeneous as respects a and b, each term being of two dimensions; so also is $5\,a\,b + 4\,a^2 - 2\,b^2$: we may therefore substitute these for a and b, provided we put the similar expressions, $2\,c^2 - 3\,c\,d + d^2$ and $5\,c\,d + 4\,c^2 - 2\,d^2$ for c and d; that is, we may infer that because

$$\frac{a}{d} = \frac{c}{d} \therefore \frac{2\,a^2 - 3\,a\,b + b^2}{5\,ab + 4\,a^2 - 2\,b^2} = \frac{2\,c^2 - 3\,c\,d + d^2}{5\,c\,d + 4\,c^2 - 2\,a^2}.$$

"The reason of this is pretty obvious; two fractions cannot be *equal* unless one is convertible into the other by multiplying numerator and denominator of the former by some factor (m); so that in the above, c must be $= m\,a$ and $d = m\,b$. Now if in the *second* of the changed fractions above you put $m\,a$, $m\,b$ for their equals c and d, you will see at once that that fraction will be nothing but the *first* fraction with its numerator and denominator multiplied by m^2. If the homogeneous expression chosen for the terms of the first fraction had been of *three* dimensions, then after the substitution of $m\,a$, $m\,b$ for c and d in the similar terms of the second fraction, the result would have

* Of like kind.

differed from the first fraction only by the numerator and denominator being multiplied by m^3, and so on, as is obvious. The particular case of this general theorem which is most frequently employed is this, namely:

$$\frac{a}{b} = \frac{c}{d} \therefore \frac{a \pm mb}{a \mp nb} = \frac{c \pm md}{c \mp nd},$$

or from $a : b :: c : d$

$$\therefore a \pm mb : a \mp nb :: c \pm md : c \mp nd,$$

where the values of m and n are arbitrary. In most applications they are chosen each equal to 1, or one equal to 0, and the other equal to 1. I need scarcely mention, that when any of the conditions of a question are expressed by a *proportion*, the product of the extremes equated to the product of the means converts the proportion into an *equation*."

We may conclude this section with an illustration of the every-day use of the principles of ratio and proportion. There are, however, so many applications of the common principle of the "rule of three" (which with too many of us is also "rule of thumb"), that we must look somewhat higher than the hackneyed but still valuable problem: "If so many men build a wall in so many days, how long will so many more men take to build the same quantity of wall?" To the scientific student a very useful application is that known as the *Chain Rule.*

This rule is employed by the computer when he wishes to ascertain the value of one unit in terms of another, whether the unit be one of commercial exchange or of physical measurement. Thus, were it required to convert

miles into inches, we should state the ratios as they follow:

$$\frac{\text{miles}}{\textit{yards}} \times \frac{\textit{yards}}{\textit{feet}} \times \frac{\textit{feet}}{\text{inches}} = \frac{\text{miles}}{\text{inches}}$$

$$\frac{1}{1760} \times \frac{1}{3} \times \frac{1}{12} = \frac{1}{63360}.$$

By cancelling the factors in *italics* (the intermediate terms) it is seen, in the first instance, that we have the required ratio, which we obtain arithmetically, in the second instance, by the rule for the multiplication of fractions (see p. 62). The ratio of a mile to an inch is as 1 : 63360.

Now, if it were further required to reduce miles per hour to metres per second, it would be necessary to state the following ratios:

$$\frac{\text{hours}}{\textit{mins.}} \times \frac{\textit{mins.}}{\text{secs.}} = \frac{\text{hours}}{\text{secs.}}$$

$$\frac{\text{miles}}{\text{inches}} \times \frac{39 \cdot 37}{1}$$

(39·37 being the number of inches per metre).

$$\frac{\frac{1}{60} \times \frac{1}{60} = \frac{1}{3600}}{\frac{1}{63360} \times \frac{39 \cdot 37}{1} = \frac{39 \cdot 37}{63360}} = \frac{63360}{39 \cdot 37 \times 3600} = \frac{63360}{141732} = \cdot 447$$

the constant required to convert miles per hour to metres per second.

Another example will show the high importance of the

principles we have studied. Suppose that the problem were put to us to determine the law which prevails between the resistances of bodies moving in the air, and their velocities. Let $V\ v$ * be any two velocities, and $R\ r$ the corresponding resistances: the problem is to find to what power of V is R proportional? Let us put x to represent this unknown power; then, if any law should hold good, we have

$$\left(\frac{v}{V}\right)^x = \frac{r}{R}.$$

According to the rules for logarithms (see Rule III., p. 23), we have

$$x \times \log. \frac{v}{V} = \log. \frac{r}{R};$$

or

$$x = \frac{\log. r - \log. R}{\log. v - \log. V};$$

or the difference of the logs. of the resistances, divided by the difference of the logs. of the velocities, will give the required power of x. In reference to the motion of projectiles and of cars on railroads, this formula occurs not unfrequently.

As a commercial example, the case of simple interest may be selected as an instance of the use of proportion. Thus:

To find the interest of a given sum, we say:

As 100 : rate per cent. : : sum : interest.

* "Many students are perplexed for a long time with such notions as that the *force* × time = velocity. They should remember that it means nothing more than this: if that which gives a feet of velocity in every second be allowed to act for b seconds, then $a\,b$ feet of velocity must result."—*De Morgan.*

For example, to find the interest on 80*l.* at 5 per cent.:

$$100 : 5 :: 80 : 4$$

To find the rate per cent., we say:

As (sum − int.) : interest :: 100 : rate per cent.

Thus:

$$\text{As } (84 - 4 = 80) : 4 :: 100 : 5$$

CHAPTER XIV.

CONTINUED PROPORTION, THE SERIES, AND THE SUMMATION OF THE SERIES.

Hitherto we have dealt with only *four* proportional quantities; but the principle of proportion can be extended to more than four (indeed to any number of) quantities. Thus we may have the following *arithmetical* proportion:

$$2, 4, 6, 8, 10, 12, 14, \text{etc.},$$

in which the *difference* is 2; or we may have the following *geometrical* proportion:

$$2, 4, 8, 16, 32, 64, 128, \text{etc.},$$

in which 2 is the common ratio.

Such a proportion, whether arithmetical or geometrical, is said to be *continued*. In fact, such a continued proportion forms an arithmetical or geometrical progression or *series*. Upon an earlier page (see p. 43) will be found the algebraic representation of these series.

The previous study of the principles governing the relation of four proportional quantities will render much

more easy our comprehension of the laws of the series, somewhat difficult to understand. We have learned that, in an arithmetical proportion, the sum of the extremes is equal to the sum of the means. This principle, widened out, also applies to a continued proportion or series. Let us take, in illustration, the seven terms of the arithmetical series given above,

$$2, 4, 6, 8, 10, 12, 14,$$

and represent the terms by the following letters,

$$a, b, c, d, e, f, g.$$

b and f or 4 and 12; c and e or 6 and 10; a and g or 2 and 14; these will now be "similar" terms (see foot-note, p. 67), because the members of each set are equally distant from the beginning or from the end of the series. The sum of $b+f$, or $4+12$, is equal to the sum of $c+e$ or of $a+g$. Whence the rule: *In any arithmetical series, or continued proportion, the sum of any two "similar" terms is equal to the sum of any other two "similar" terms.*

When the number of terms is odd, there will be a middle term; twice this middle term (as above, $2d$) will be equal to the sum of any two "similar" terms of the series. This principle will guide us to a simple rule for finding the *sum* of *any* number of terms of an arithmetical series. We have seen that $b+f$, $c+e$, and $a+g$ are equal, and that d, the middle term, is equal to half the sum of any two similar terms, whence we can at once perceive that the sum of the terms, or $(a+g)+(b+f)+(c+e)+d$, is equal to $3\frac{1}{2}$ times $(a+g)$, or $3\frac{1}{2}$ times the sum of the extremes, or first and last terms. There are seven terms; $3\frac{1}{2}$ is equal to $\frac{7}{2}$, or half the number of terms. We have

then the rule: *To sum any number of terms of an arithmetical series, multiply the sum of first and last terms (or the sum of any two "similar" terms) by half the number of terms.*

If the example selected for illustration had contained an even number of terms, say of six terms, a, b, c, d, e, f, we should have had $a + f = b + e = c + d$, and we perceive that the sum of the series is three times $a + f$, or, as before, the sum of two similar terms multiplied by half the number of terms.

In a former page, the following formula has been given to find the last term (l) where that term is not stated. The formula is

$$l = a + (n - 1)\,d,$$

where l is the last term, a the first term, n the number of terms, and d the common difference. This formula, translated into words (it supposes the first term given, and also the common difference and number of terms of the continued proportion), tells us "that the last term may be found by adding to the first term (a) the common difference (d) multiplied by one less than the number of terms ($n - 1$). Professor De Morgan gives the reason of this process in these words: "The passage from the first to the nth term is made by $n - 1$ steps, at each of which the common difference is added."

The remaining problem is to find the common difference when the sum, the number of terms, and the first term of an arithmetical series are given. To obtain a solution, we must first find the last term. This we get from the knowledge that the sum is the result of multiplying the sum of the first and last terms by half the number of terms, consequently the sum *divided* by half the number of terms will

give a quotient which minus the first term is equal to the last term of the series; or

$$l = \frac{\frac{s}{n}}{2} - a.$$

Now we have to pass from a, the first term, to l, the last term, by $n - 1$ equal steps (the first step having been made at the first term, and n being the number of terms). Therefore the common difference

$$d = \frac{l - a}{n - 1}$$

or the ratio of the last term minus the first term to the number of terms less one. For example: Let the first term of an arithmetical progression be 3, the number of terms 50, and the sum 2600, to find the last term and the common difference.

The sum 2600 divided by half the number of terms $\left(\frac{50}{2} =\right)$ 25 is equal to 104, and this less the first term $(101 - 3 = 98)$, divided by the number of terms less one $(50 - 1) = 2$ the common difference, the series is therefore

$$3, 5, 7, 9, 11 \ldots\ldots 101.$$

Similar short rules and simple reasons may be found for the processes of geometrical progression, or, in mathematical language, for the summation of the geometrical series.

If r be the common ratio, the geometrical series may be represented by the general algebraic formula

$$a,\ a\,r,\ (a\,r)\,r,\ (a\,r\,r)\,r, \text{ etc., or}$$
$$a,\ a\,r,\ a\,r^2,\ a\,r^3, \text{ etc.}$$

If, as in the example before given, $r = 2$, and a the first term is 2, we have

$$2, 4, 8, 16, 32, \text{etc.}$$

The rule quoted with regard to the *sum* of similar terms of *arithmetical* series, will of course hold good with a *geometrical* series if for the word *sum* we substitute *product*. The product of any two "similar" terms of a geometrical series is equal to the *product* of any other two "similar" terms of the same series. If the number of terms be *odd*, the *product* of any two "similar" terms will be equal to the *square of the middle term* (an expansion of the principle with which we became acquainted when considering the proportion of only four quantities).

When it is considered that the nth term of a geometrical series is nothing more than the multiplication of the first term by the common ratio raised to the $(n-1)$th power, it is easily seen that

$$l = a\,r^{n-1};$$

and that it is necessary only to know the first term and common ratio to write down the succeeding terms to any number.

To learn how to sum any number of terms of a continued geometrical proportion, we must look again at the series a, $a\,r$, $a\,r^2$, $a\,r^3$. The sum of this, or $a + a\,r + a\,r^2 + a\,r^3$ is evidently equal to $a\,(1 + r + r^2 + r^3)$; and this again is equal to

$$a \times \frac{r^4-1}{r-1} = \frac{a\,r^4-a}{r-1}, \qquad (1)^*$$

* That $\dfrac{r^{(n+1)}-1}{r-1} = 1 + r \ldots . + r^n$ is capable of proof by the following reasoning. For example, take

$$\frac{r^3-1}{r-1} = 1 + r + r^2. \quad r^3 - 1 = r^3 - r^2 + r^2 - r + r - 1,$$

when r is greater than 1; or equal to

$$a \times \frac{1-r^4}{1-r} = \frac{a-ar^4}{1-r},$$

when r is less than 1.

From equation (1) we obtain the following general expression for the sum of the series, that is,

$$S = \frac{ar^n - a}{r-1}.$$

Again, since $l = ar^{n-1}$, we have $lr = ar^n$; and as we may substitute equivalent values in any equation, the following expression is obtained from that first given,

$$S = \frac{rl-a}{r-1}.$$

This, in words, means that to sum the series, the last term is to be multiplied by the common ratio, the first term is to be taken from the product; the result to be divided by the ratio less 1.

since when we subtract r or its power we subsequently add r or the same power. $r^3 - r^2$ is obviously equal to $r^2(r-1)$, and $r^2 - r$ is equal $r(r-1)$, whence $r^3 - r^2 + r^2 - r + r - 1 = r^2(r-1) + r(r-1) + r - 1$. Again, $r^2(r-1) + r(r-1) + r - 1$ is $r^2 + r + 1$ taken $r-1$ times. In the second case, where r is less than unity, $1 + r + r^2$ multiplied by $1 - r$ is equal to $1 - r + r(1-r) + r^2(1-r)$, and this equals

$$1 - r + r - r^2 + r^2 - r^3,$$

which again is equal to $1 - r^3$. As $1 - r^3$ is obtained by multiplying $1 + r + r^2$ by $1 - r$, so dividing $1 - r^3$ by $1 - r$ must yield the quotient, $1 + r + r^2$. r^3 is a term higher $(n+1)$ than r^2; hence the rule: "To find the sum of n terms of the series $1 + r + r^2 +$ etc., divide the difference of one and the $(n+1)$th term by the difference between 1 and r."

When r is less than 1, or is fractional, we have, according to the preceding formulæ,

$$S = \frac{a - r\,l}{1 - r}, \text{ or } = \frac{a - a\,r^n}{1 - r}.$$

In the latter case, the series is a decreasing one, and its terms continually approach 0 in value. Could the series be continued to *infinity* we should undoubtedly arrive at 0; that is, the last term (l) would be $= 0$. The above equation therefore becomes (since multiplying by 0 gives 0)

$$\Sigma^{*} = \frac{a}{1 - r},$$

the sum of an infinite series, or when $n = \infty$.†

Examples

Of Increasing Geometrical Progression.—First term $= 1$; ratio $= 2$. Required the tenth term and the sum. Answer: $l = a\,r^{n-1} = 1 \times 2^9 = 512$, the tenth term. $S = \frac{r\,l - a}{r - 1} = \frac{1024 - 1}{1} = 1023$, the sum.

Of Decreasing Series.—Sum the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$, to seven terms (1), and also to infinity (2). Answer: (1)

$$r = \tfrac{1}{2},\ n = 7,\ l = a\,r^{n-1} = (\tfrac{1}{2})^6 = \tfrac{1}{64}.$$

$$S = \frac{a - l\,r}{1 - r} = \frac{1 - \frac{1}{128}}{1 - \frac{1}{2}} = \frac{\frac{127}{128}}{\frac{1}{2}} = \tfrac{127}{128} \times 2 = 1\tfrac{63}{64},$$

the sum of seven terms. Answer: (2)

$$\Sigma = \frac{a}{1 - r} = \frac{1}{1 - \frac{1}{2}} = 2, \text{ the sum to infinity.}$$

* Σ (the Greek capital *sigma*) is generally employed to represent the sum of an infinite series.

† The character ∞ is mathematically employed to represent infinity.

Find the vulgar fraction equivalent to the circulating decimal, $\cdot 363636$. Answer: This decimal expressed as a series is $\frac{36}{100} + \frac{36}{1000} + \frac{36}{10000} +$ etc., the first term being $\frac{36}{100}$ and the ratio $= \frac{1}{100}$. Therefore,

$$\Sigma = \frac{n}{1-r} = \frac{\frac{36}{100}}{\frac{99}{100}} = \frac{4}{11}.$$

Geometrical Means.

Professor Young, treating this point very simply, says: "When the leading term and any remote term are given, we may always find the intervening terms, and thus fill up the gap; for instance, if a and $a\,r^4$ be given, to find the three intervening terms we should divide $a\,r^4$ by a; and knowing—from the fact that there are *five* terms altogether—that the quotient would be the *fourth* power of the ratio, we should obtain the ratio itself by taking the fourth root of that quotient: by help of the ratio, the wanting terms may be easily supplied. These intervening terms are called *geometrical means* between the given extremes; if only *one* mean is to be inserted between two extremes, it is found by taking the square root of the product of the extremes. Suppose, for example, it were required to find a geometrical mean between 2 and 8: then, since the square root of 8×2, or 16, is 4, we know the mean to be this number; the progression being 2, 4, 8: if the proposed extremes had been -2, -8, we should have taken the *minus* root of 16, and have written the progression thus, -2, -4 -8: nevertheless, whether the given extremes be both *plus* or both *minus*, the square root of their product with *either* sign may be truly regarded as a *mean*, for the following are geometrical progressions, as well as those above, 2, -4, 8; -2,

4, -8; the common ratio *here* being -2; in the sets above it was 2. Again, suppose we had to insert two geometrical means between 3 and 81; then, as there are *four* terms altogether, the exponent $n-1$ above, is here 3, $\therefore 81 \div 3 = 27 = r^3 \therefore r = 3$. Hence the progression is 3, 9, 27, 81; the required *means* being 9 and 27."

CHAPTER XV.

LIMIT OF SERIES.

THE powers of a quantity greater than unity, increase without limit. Thus, there is no power of 2 but that the next higher power is greater. Improper fractions follow, of course, the same law; thus, $1\frac{1}{2}$ raised to the second power, or $1\frac{1}{2} \times 1\frac{1}{2}$ as it contains the half of one-and-a-half more than one-and-a-half, is greater than the first power. The powers of unity only never increase. And the powers of a proper fraction, or of a quantity less than unity, always decrease. Thus the powers of $\frac{1}{2}$, viz. $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, etc., constantly diminish. Representing an integral quantity by x and a fractional quantity by $\frac{1}{x}$, the powers of the first, or x, x^2, x^3, etc., continuously increase, while the powers of $\frac{1}{x}$, viz. $\frac{1}{x^2}$, $\frac{1}{x^3}$, etc., diminish in value. Thus, with the series

$$x, x^2, x^3 \ldots . x^n, \text{ etc.},$$

we have the following conditions:

I. An increasing series, if x is greater than unity.

II. A decreasing series, if x is less than unity.

III. A series whose terms are all of the same value, when $x = 1$.

In the first and third cases, the sum of the series may evidently be made as great as we please by the addition of more terms. But where x is less than unity, this may or may not be possible.

If we take the decreasing series,

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32}, \text{ etc.},$$

the *sum* of this series will constantly approach 2 in value, but will never attain that value. It will always be necessary to add the last term to obtain the value 2. Thus:

$$(1 + \tfrac{1}{2}) + \tfrac{1}{2} = 2.$$
$$(1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8}) + \tfrac{1}{8} = 2.$$
$$(1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32}) + \tfrac{1}{32} = 2.$$

2 is, then, the limit towards which the series

$$1 + \tfrac{1}{2} + \tfrac{1}{4}, \text{ etc.},$$

constantly approaches.*

Such a series as $1, r, r^2, r^3, \ldots . r^n$ has, then, *always* a limit when x is less than unity. The powers of r are constantly decreasing in value, and the higher the name of the power, the lower its value. Let n be a very distant term; $\frac{1-n}{1-r}$ (see p. 78) will be the sum. But

$$\frac{1-n}{1-r} = \frac{1}{1-r} - \frac{n}{1-r},$$

whence we see that the more distant is the term n (or, rather, the smaller the fraction $\frac{n}{1-r}$), the less is $\frac{1}{1-r}$

* But we are not therefore to conclude that *every* decreasing series has a limit. It is possible to arrange a series (for instance, the reciprocals of the integer numbers in lots, each containing half as many terms as there are units in the denominator of its last term) having no limit.

affected by the subtraction. $\frac{1}{1-r}$ is then the *limit* towards which the series $1 + r + r^2$, etc., approaches.

The formula given in our last section for the summation of an infinite series was

$$\Sigma = \frac{a}{1-r},$$

where a is the first term. Substitute 1 for a, and we have as well by this method the expression

$$\frac{1}{1-r} = 1 + r \ldots\ldots r^n,$$

where $n = \infty$. Whence we perceive that "the sum of an infinite series is the limit towards which we approximate by continually adding more and more of its terms."

CHAPTER XVI.

SQUARE AND CUBE ROOTS.

We have learned that a quantity multiplied into itself is said to be "*squared*"; when multiplied twice into itself said to be "*cubed*," &c. And we know that the quantity so multiplied is said to be the *root* of the resulting square or cube.

The *square* root of a quantity is, then, that other quantity which multiplied into itself will produce the first-mentioned quantity. Thus the square root of 16 is 4, because 4 multiplied into itself produces 16.

In algebraic symbols the sign of the root is, we have learned, $\sqrt{\ }$ or $\surd$. Thus $\sqrt{16} = 4$.

The cube root of a quantity is that other quantity which

multiplied twice into itself will produce the first-mentioned quantity. The symbol for the cube root is $\sqrt[3]{\ }$; and similarly the symbol for any root is the radix sign $\sqrt{\ }$ carrying in its lap the integer indicating the root required. Thus the symbol for the nth* root is $\sqrt[n]{\ }$.

Roots are also represented by fractional indices; as $a^{\frac{1}{2}}$ for $\sqrt{a}$, and $x^{\frac{1}{3}}$ for $\sqrt[3]{x}$. And generally $x^{\frac{1}{n}} = \sqrt[n]{x}$.

With regard to roots, the following are self-evident propositions:

1. That the nth root of a quantity multiplied by the nth root of the same quantity is the quantity itself.

$$\sqrt[n]{a} \times \sqrt[n]{a} = a\text{; or, for instance, } \sqrt{4} \times \sqrt{4} = 4.$$

2. That the nth root of the nth power of the quantity is the quantity itself. $\sqrt[n]{a^n} = a$. $\sqrt[3]{x^3} = 2$.

3. That the nth root of a quantity is equal to the *product* of the nth roots of its factors. $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$; for instance,

$$\sqrt[2]{4 \times 9} = \sqrt{36} = 6, \text{ and } \sqrt{4} \times \sqrt{9} = 2 \times 3 = 6.$$

To find the nth root of a monomial: The nth root of the coefficient (if any) must be found for the coefficient of answer. Then, divide the exponent of each quantity by the radical number (or, as it is termed, the exponent of the radical). Thus,

$$\sqrt{16\,x^6} = 4\,x^3, \text{ because } \sqrt{16} = 4 \text{ and } x^{\frac{6}{2}} = x^3.$$

As to the sign to be prefixed to the quantity obtained, it is clear (if we consider the principle that the multiplication or division of quantities with like signs produces positive (+) quantities, and of quantities with unlike signs nega-

* n is commonly chosen to represent, instead of saying, any number.

tive ($-$) quantities) that the *odd* root of any quantity has the *same* sign as that quantity, but that the *even* root of a *positive* quantity may be positive or negative. This indefinite condition as to sign is indicated by the double sign $\pm$. As the square root of a quantity is that which when squared produces the quantity, it is also clear, upon the expenditure of a little thought, that the *even* root of a *negative* quantity is impossible. Algebra, however, admits of such expressions as $\sqrt{-x}$, these expressions being termed *imaginary*. For their influence and their true meaning the reader should consult a work of wider scope than this can naturally be.

There are quantities whose exact root cannot be determined; as $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, since these quantities, 3, 5, 7 are not squares. Such quantities are termed *irrational* or *surds*. But by proposition 3 just given we may simplify the aspect of a *surd* quantity; thus,

$$\sqrt{8} = \sqrt{4} \times \sqrt{2} = 2\sqrt{2}.$$

The question of finding the root of a polynomial (a quantity of more than one term, see p. 48) is somewhat complicated. The square of $(a + b)$, or $(a + b)^2$ is $a^2 + 2ab + b^2$, or the squares of the two quantities a and b, together with twice the product of a and b. Let it be required to find the square root of $(a^2 + 2ab + b^2)$; we know that it is $(a + b)$, but the knowledge will help us to learn the process usually employed in finding the square root.

Arranging the terms as for division, we have

$$\begin{array}{r|l} & a^2 + 2ab + b^2 \quad (a + b \\ & \underline{a^2 \qquad\qquad\quad} \\ 2a + b] & \qquad\quad 2ab + b^2 \\ & \qquad\quad \underline{2ab + b^2} \end{array}$$

and we may derive the following rule: Take the square root of the leading term (in the above it is a) and place it as quotient; the square of this (a^2) is placed beneath the leading term, to which it will be equal. Bring the next *two* terms ($2\,a\,b + b^2$ above) for a dividend, leaving a place for a divisor (as where $2\,a + b$ in the above example is placed) to the left. Take twice the term placed in the quotient (in the example as far as we are now supposed to have worked we have a in the quotient, hence we take $2\,a$) and ascertain how often this *incomplete* divisor is contained in the leading term of the dividend (the leading term above is $2\,a\,b$) and add the quotient with its sign (the quotient of $2\,a\,b$ by $2\,a$ is b) to the quotient (a) already existing, and to the incomplete divisor ($2\,a$). Then proceed as in the division of compound quantities.

If there be a remainder, the next two terms are to be brought down to form another dividend, and the proceedings repeated until the terms are exhausted.

If there should be a remainder when the terms are exhausted, the quantity treated is a surd; and we learn that the polynomial could be made a *complete square* by subtracting the remainder from it.

The following is another example of the finding of the root of a *complete* square:

$$
\begin{array}{rl}
 & x^4 - 2\,x^3 + \tfrac{3}{2}\,x^2 - \dfrac{x}{2} + \tfrac{1}{16}\ (x^2 - x + \tfrac{1}{4} \\
 & \underline{x^4} \\
2\,x^2 - x) & \quad - 2\,x^3 + \tfrac{3}{2}\,x^2 \\
 & \quad \underline{- 2\,x^3 + \quad x^2} \\
2\,x^2 - 2\,x + \tfrac{1}{4}) & \qquad + \tfrac{1}{2}\,x^2 - \dfrac{x}{2} + \tfrac{1}{16} \\
 & \qquad \underline{+ \tfrac{1}{2}\,x^2 - \dfrac{x}{2} + \tfrac{1}{16}}
\end{array}
$$

We can now advance from these particular examples to a more general rule for the extraction of higher roots,* which rule applies to the finding of cube roots and higher roots: the student can form examples for himself by selecting powers of the binomials given on page 51, or of any other expressions of which he knows the root, and applying the instructions step by step as follows:

1. Place the root of the first or leading term in the quotient.
2. Subtract the power, and
3. Bring down next term for dividend.
4. Raise the root found in (1) to the next lower power than in (2), and multiply it by the index of the power in (2).
5. Divide (3) by the result of (4); the quotient will be the next term of the root.
6. Involve the whole, subtract and divide, and proceed until complete.

If higher roots are to be taken, the roots of roots may be extracted: thus for the fourth root we may extract the square root of the square root; for the sixth root, the cube root of the square root; for the eighth root, the square root of the square root of the square root; for the ninth root the cube root of the cube root.

In practice, however, it may be said that such high roots rarely occur. But it would be easy if the example were numerical and not algebraic to effect the reduction by means of logarithms, or by means of the following formula for the nth root.

Let A be the given number of which we wish to find the nth root, n being the index of the root, and let x be

* Because the student will certainly not attempt the extraction of higher roots until his understanding of the principles previously laid down will admit of his perception of the sequence.

an approximation to the root; then a closer approximation (which may be carried to as nearly the truth as we please) is given by

$$\frac{(n+1)\,A + (n-1)\,x^n}{(n-1)\,A + (n+1)\,x^n} \times x.$$

As an example, let it be required to extract the cube root of a number A. We have then the index n equal to 3, so that the formula becomes

$$\frac{(3+1)\,A + (3-1)\,x^3}{(3-1)\,A + (n+1)\,x^3} \times x,$$

and this is equivalent to

$$\frac{4\,A + 2\,x^3}{2\,A + 4\,x^3} \times x.$$

We now substitute for A the given number, and for x the approximation to the required root. The result of the process will be either the true root or an approximation to it; and if it be the latter, we can obtain a still closer approximation by substituting for x the approximation we have obtained.*

* The computation of arithmetical cube roots of numbers not exceeding six or nine figures has often been considered a feat in mental arithmetic, and it may not be uninteresting if we here suggest the means of accomplishment. The cube of any number consisting of two digits must be a number of not less than four and of not more than six digits; the least, the cube of 10 is 1000 containing four digits, and the greatest, the cube of 99 is 970,299, which contains six digits. Now let the number whose cube root is to be extracted, be divided into two parts, the right-hand portion containing always three digits, and the left-hand portion one, two, or three digits, as the case may be. And let the computer commit to memory the following cubes of the nine digits:

CHAPTER XVII.

EQUATIONS.

In the preceding pages it has been shown how algebraic operations are represented in symbols, and that these algebraic symbols are combined in certain arbitrary forms and ways to indicate shortly and without words the processes required to obtain a certain result. This result is known when those operations are carried out; and we may represent the result by a letter which is placed on one side of the sign of equality (=), the operations being indicated by the symbols on the other side of the sign =. Such a representation of equality or of the means of obtaining equality is, we have learned (p. 3), termed an equation.

An equation is termed *simple* when it contains x only as a first power (x itself); when the equation contains x^2 it is termed a *quadratic* equation (if x is present as a first as well as in the second power the quadratic equation is

---

No.	Cube.	No.	Cube.	No.	Cube.
1	1	4	64	7	343
2	8	5	125	8	512
3	27	6	216	9	729

Then finding which of the nine digits will give a number next less than that represented by the left-hand portion of the number whose cube root is to be found, this digit will be the first or tens' figure of the required root, and that digit giving the units' figure of the right-hand portion will be the units' figure of the root. For example, let it be required to find the cube root of 551,368. The next nearest less cube to 551 is 512 or 8^3; and to the portion 368 the only digit giving when cubed the units' figure, viz. 8, is 2. The root required is 82. In a somewhat similar manner operations can be extended to nine digits.

said to be *adfected* or *affected*; $x^2 = a$ is a simple quadratic, $x^2 = a\,x$ an adfected quadratic); an equation containing the fourth power of x (or x^4) is termed a *biquadratic* equation; and when x^3 is present as the highest power the equation is said to be *cubic*.

We have first to deal with the reduction and solution of *simple* equations.

The reduction of an equation consists in so disengaging the unknown quantity (p. 3) that it may stand alone; thus if from the equation $x + m = n$ we wish to find the value of x, we must take away m from the member* $(x + m)$, and we must also take away the value of m from the member n, or the n would be too great by the value of m. The equation then stands

$$x + m - m = n - m,$$

and as $+ m - m = 0$, we have

$$x = m - n.$$

But in the reduction of equations in practice, as it would be cumbrous to go through the process of performing the same operation on both members of the equation, the following rules are adopted:

If a known quantity is joined by the sign of addition to the unknown quantity, this known quantity is transposed to the other member of the equation with its sign changed. Thus $x + a = c$ gives $x = c - a$.

And conversely, a quantity joined by the sign of subtraction to the unknown quantity is transposed to the other member with its sign changed to that of addition; or $x - a = c$ gives $x = c + a$.

* The sign = is said to separate the *members* of the equation.

If the unknown quantity is divided by a known quantity, the known quantity is transposed to the other term as a multiplier. Thus $\frac{x}{a} = c$ gives $x = a\,c$.

If the unknown quantity is multiplied by a known quantity, this known quantity is transposed to the other member as a divisor; thus $a\,x = c$ gives $x = \frac{c}{a}$.

Again, $\sqrt{x} = a$ gives $x = a^2$, and $x^2 = a$ gives $x = \sqrt{a}$.

In a word, the opposite operations are performed on the two members.

When fractional quantities occur in an equation, they may be removed by multiplying both members of the equation by the denominator of the fraction to be removed. Thus,

$$\frac{5\,a}{6} = \frac{x}{3}^{*} \text{ gives } \left(\frac{5\,a}{\cancel{6}}\right) \times \cancel{6} = \left(\frac{x}{3}\right) \times 6,$$

and this is equal to $5\,a = \frac{6\,x}{3}$.

Also

$$(5\,a) \times 3 = \left(\frac{6\,x}{\cancel{3}}\right) \times \cancel{3},$$

or

$$15\,a = 6\,x,$$

which is the expression $\frac{5\,a}{6} = \frac{x}{3}$ cleared of fractions.

* The expression $\frac{5}{6}\,a$ is obviously equivalent to $\frac{5\,a}{6}$; and $\frac{x}{3}$ to $\frac{1}{3}\,x$.

Equations with more than one unknown quantity.

Equations may and often do contain more than one unknown quantity. When this is the case, these unknown quantities are determined by a series of independent equations. The student should note that there must be *as many equations* as there are *unknown quantities*, or solution cannot be made. That this condition is satisfied should be the first inquiry, because work expended under any other condition would be wasted.

In cases such as the following, the equation can be solved by applying the principles already explained. For instance, let $xy = a$ and $\frac{x}{y} = b$; then $xy \times \frac{x}{y} = \frac{x^2 \cancel{y}}{\cancel{y}}$ or $x^2 = ab$, whence $x = \sqrt{ab}$; and from this the value of y $\left(\text{or } y = \frac{a}{x}\right)$ can be determined.

Generally other cases can be reduced to one of the following forms:

With two unknowns, x and y:

$$\text{If } x + y = s \quad \text{and} \quad x - y = d,$$

$$x = \frac{s + d}{2} \quad \text{and} \quad y = \frac{s - d}{2}.$$

$$\begin{aligned} \text{If } a_1 x + b_1 y &= c_1, \text{ and} \\ a_2 x + b_2 y &= c_2, \end{aligned}$$

$$x = \frac{c_1 b_2 - c_2 b_1}{a_1 b_2 - a_2 b_1} \qquad y = \frac{c_1 a_2 - c_2 a_1}{b_1 a_2 - b_2 a_1}.$$

$$\begin{aligned} \text{If } a_1 x + b_1 y + c_1 z &= d_1 \\ a_2 x + b_2 y + c_2 z &= d_2 \\ a_3 x + b_3 y + c_3 z &= d_3 \end{aligned}$$

$$x = \frac{d_1(b_2c_3 - b_3c_2) + d_2(b_3c_1 - b_1c_3) + d_3(b_1c_2 - b_2c_1)}{a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)}$$

$$y = \frac{d_1(a_2c_3 - a_3c_2) + d_2(a_3c_1 - a_1c_3) + d_3(a_1c_2 - a_2c_1)}{b_1(a_2c_3 - a_3c_2) + b_2(a_3c_1 - a_1c_3) + b_3(a_1c_2 - a_2c_1)}$$

$$z = \frac{d_1(a_2b_3 - a_3b_2) + d_2(a_3b_1 - a_1b_3) + d_3(a_1b_2 - a_2b_1)}{c_1(a_2b_3 - a_3b_2) + c_2(a_3b_1 - a_1b_3) + c_3(a_1b_2 - a_2b_1)}$$

Quadratic Equations.

These are solvable by one of the forms of operation given in the appended list of equational processes at the end of this book; but it is necessary to remark that a quadratic equation may have *two* root-results. This arises from the fact that a positive quantity multiplied into itself gives a positive product, and that a negative quantity multiplied into itself also gives a positive product. So that x^2 may have two roots, namely, $+x$ or $-x$; for instance, $\sqrt{4}$ may be -2 or $+2$, since $(-2) \times (-2)$ is 4 just as $(+2) \times (+2)$ is 4.

The student should now be able either to work out any technical formulæ he may wish, or to take in hand with advantage and understanding any one of those authors who have dealt more fully with the subject than it has been the purport of this work to do.

APPENDIX.

A List of Formulæ embodying the Rules and Principles explained in the preceding pages, or that may be deduced therefrom.

Addition and Subtraction:

$$n + (-m) = n - m = -(m - n)$$
$$n - (+m) = n + (-m)$$
$$n - (-m) = n + (+m) = n + m.$$

Multiplication:

$$a \,.\, b = a \times b = ab$$
$$n(m + p) = nm + np$$
$$(-n) \times (-m) = +mn$$
$$(+n)(-m) = -mn.$$

Division:

$$a \div b = \frac{a}{b} = a : b$$

$$(-a) : (-b) = a : b = +\frac{a}{b}$$

$$(-a) : (+b) = (+a) : (-b) = -\frac{a}{b}$$

$$\frac{m \pm n}{p} = \frac{m}{p} \pm \frac{n}{p}$$

$$\frac{mn}{p} = \frac{m}{p} \cdot n = m \cdot \frac{n}{p}$$

$$\frac{p}{mn} = \frac{p}{m} : n = \frac{p}{n} : m.$$

Fractions:

$$\frac{a}{b} = \frac{an}{bn} = \frac{a : n}{b : n}$$

$$\frac{a}{b} \pm \frac{n}{b} = \frac{a \pm n}{b}$$

$$\frac{a}{b} \cdot \frac{n}{p} = \frac{an}{bp}$$

$$\frac{a}{b} : \frac{n}{p} = \frac{a}{b} \cdot \frac{p}{n} = \frac{ap}{bn}.$$

Powers and Roots:

$$a^m \cdot b^m = (ab)^m$$

$$a^m : b^m = \left(\frac{a}{b}\right)^m$$

$$a^m \cdot a^n = a^{m+n}$$

$$a^m : a^n = a^{m-n}$$

$$(a^m)^n = a^{mn}$$

$$(-a)^{2n} = a^{2n}$$

$$(-a)^{2n+1} = -a^{2n+1}$$

$$(a \pm b)^2 = a^2 \pm 2ab + b^2$$

$$(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$$

$$n^0 = 1 \qquad x^{-m} = \frac{1}{x^m} \qquad x^m = \frac{1}{x^{-m}}$$

$$a^{-\infty} = \text{either } 0 \text{ or } \infty$$

$$a^{\frac{1}{m}} = \sqrt[m]{a}$$

$$a^{m} = \sqrt[\frac{1}{m}]{a}$$

$$\sqrt[m]{a} \,.\, \sqrt[m]{b} = \sqrt[m]{a\,b}$$

$$\sqrt[n]{a} : \sqrt[n]{b} = \sqrt[n]{\frac{a}{b}}$$

$$\sqrt[m]{a^{n}} = (\sqrt[m]{a})^{n}$$

$$\sqrt[n]{\sqrt[m]{a}} = \sqrt[m]{\sqrt[n]{a}}$$

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[m\,n]{a}$$

$$\sqrt[2m]{+\,a} = \pm\, b$$

$$\sqrt[2m]{-\,a} = \pm\, b\,\sqrt{-1}$$

$$\sqrt[2m+1]{+\,a} = +\,c$$

$$\sqrt[2m+1]{-\,a} = -\,c.$$

Logarithms:

$$\text{log. } m\,n = \text{log. } m + \text{log. } n$$

$$\text{log. } \frac{m}{n} = \text{log. } m - \text{log. } n$$

$$\text{log. } x^{m} = m \text{ log. } x$$

$$\text{log. } \sqrt[m]{x} = \frac{\text{log. } x}{m}.$$

Equations:

Let $x \pm m = n$, **then** $x = n \mp m.$

Let $n\,x = m$, **then** $x = \frac{m}{n}.$

Let $\frac{x}{n} = m$, then $x = n\,m$.

Let $\frac{n}{x} = m$, then $x = \frac{n}{m}$.

Let $x^n = m$, then $x = \sqrt[n]{m}$.

Let $\sqrt[n]{x} = m$, then $x = m^n$.

Let $a^x = b$, then $x \log. a = \log. b$, and $x = \frac{\log. b}{\log. a}$.

$x : n = m : p$, or $\frac{x}{n} = \frac{m}{p}$, then $x\,p = n\,m$.

$x : n = m : x$, or $n : x = x : m$ gives $x^2 = m\,n$ and $x = \sqrt{m\,n}$.

$x : n = m : p$ gives $x : m = n : p$.

$x : n = m : p$, and $(x \pm n) : n = (m \pm p) : p$.

Quadratic Equations:

$x^2 + a\,x = b$ gives

$$x = -\frac{a}{2} \pm \sqrt{b + \left(\frac{a}{2}\right)^2};$$

$x^{2n} + a\,x^n = b$ gives

$$x = \sqrt[n]{-\frac{a}{2} \pm \sqrt{b + \left(\frac{a}{2}\right)^2}};$$

$x + y = s$, and $x\,y = p$; then

$$x = \frac{s + \sqrt{s^2 - 4\,p}}{2}; \text{ and } y = \frac{s - \sqrt{s^2 - 4\,p}}{2}.$$

Cubic Equation:

The quadrinomial cubic equation

$$x^3 + a x^2 + b x + c = 0$$

is converted into a trinomial

$$x_1{}^3 + b_1 x_1 + c_1 = 0$$

if we put

$$x_1 = x - \frac{a}{3}, \quad b_1 = b - \frac{a^2}{3}, \quad c_1 = c - \frac{a b}{3} + \frac{2}{27} a^3.$$

Cardan's rule for the solution of the cubic equation of the form $x^3 + b x + c = 0$ is

$$x = \sqrt[3]{-\frac{c}{2} + \sqrt{\left(\frac{b}{3}\right)^3 + \left(\frac{c}{2}\right)^2}}$$

$$+ \sqrt[3]{-\frac{c}{2} - \sqrt{\left(\frac{b}{3}\right)^3 + \left(\frac{c}{2}\right)^2}}.$$

This rule furnishes a true result if b is positive, or if b is negative and $\left(\frac{b}{3}\right)^3$ is greater than $\left(\frac{c}{2}\right)^2$.

If b is negative and $\left(\frac{b}{3}\right)^3 = \left(\frac{c}{2}\right)^2$, the equation has the following three true roots:

$$x = -2\sqrt[3]{\frac{c}{2}}, \quad x = \sqrt[3]{\frac{c}{2}}, \quad x = \sqrt[3]{\frac{c}{2}}.$$

If b is negative, and $\left(\frac{b}{3}\right)^3$ is greater than $\left(\frac{c}{2}\right)^2$, the roots are real, but imaginary.

Approximation Formulæ:

If x_1 is an approximation to $x^2 + ax + b = 0$, then

$$x \text{ (nearly)} = \frac{x_1^2 - b}{2x_1 + a}.$$

If x_1 is an approximation to $x^3 + ax^2 + bx + c = 0$, then

$$x = \frac{2x_1^3 + ax_1^2 - c}{3x_1^2 + 2ax_1 + b}.$$

If x_1 is an approximation to $x^4 + ax^3 + bx^2 + cx + d = 0$, then

$$x = \frac{3x_1^4 + 2ax_1^3 + bx_1^2 - d}{4x_1^3 + 3ax_1^2 + 2bx_1 + c}.$$

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